Projective State Spaces for Quantum Gravity

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Introduction

Quantum Field Theory without a Vacuum State

Standard approach to QFT:

- discrete excitations around a **vacuum** state
- $\neq$ vacuum states $\leftrightarrow$ **inequivalent** representations $\Rightarrow$ need to fine-tune the vacuum state to the **dynamics**
  - ↳ QFT on curved space time: no natural choice
  - ↳ LQG: preferred kinematical vacuum [Lewandowski, Okołów, Sahlmann, Thiemann; Fleischhack] but tension with the dynamics and the semi-classical limit [see also: Koslowski & Sahlmann '11]

An alternative way to construct the state space [Kijowski '76, Okołów '09 & '13], by gluing together elementary **building blocs**:

- **lifting** the Stone-von Neumann theorem to QFT
- yields **universal** quantum state spaces, independent of any choice of **polarization**

Contents

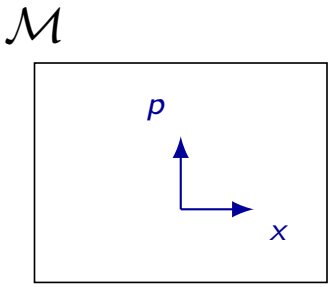
Projective Formalism for QFT
- Building Blocs: Finitely many Degrees of Freedom
- Coarse-graining between Phase Spaces
- Coarse-graining between Quantum State Spaces
- Natural Embedding of Arbitrary Vacuum Sectors

Universal Selections of Degrees of Freedom

Application to Loop Quantum Gravity

Projective Formalism for QFT

Building Blocs: Finitely many Degrees of Freedom



$$\mathsf{a} = \frac{\hat{x} + i\,\hat{p}}{\sqrt{2}}$$

$$|n\rangle = \frac{\mathsf{a}^{\dagger n}}{\sqrt{n!}}\,|\varnothing\rangle$$

Example: linear system

- modes = **independant** pairs of canonically **conjugate** variables
- polarization $\rightsquigarrow$ Fock representation

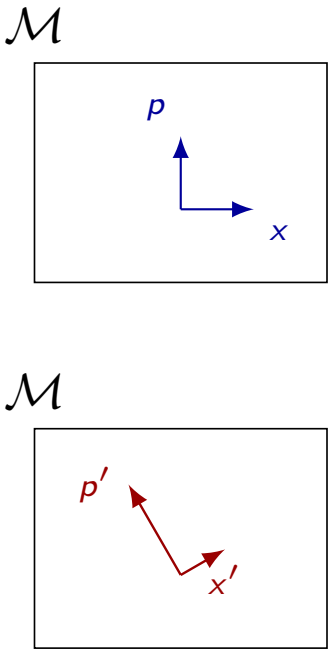
Relating different choices:

- finite dim $\Rightarrow$ unitarily equivalent representations (but $\neq$ vacuum states)
- infinite dim $\Rightarrow$ in general, **not** equivalent

[Uniqueness for finite dim linear: Stone, von Neumann; Geometric quantization: Woodhouse,...]

Projective Formalism for QFT

Building Blocs: Finitely many Degrees of Freedom



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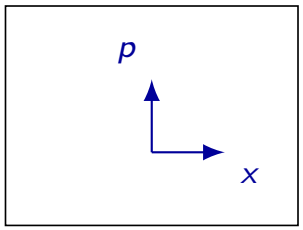
Projective Formalism for QFT

Building Blocs: Finitely many Degrees of Freedom

$\mathcal{M}$



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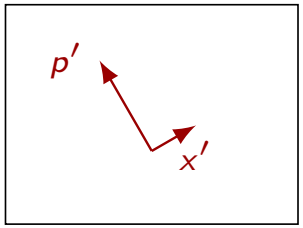

$$\mathsf{a}' = \alpha\,\mathsf{a} + \beta\,\mathsf{a}^\dagger$$

$$|\varnothing'\rangle = \sum_p \xi_{2p}\,|2p\rangle$$

$|\alpha|^2 - |\beta|^2 = 1$, $\xi_{2p} := \left(1 - (\beta/\alpha)^2\right)^{1/4} \frac{\sqrt{(2p)!}}{p!} \left(-\frac{\beta}{2\alpha}\right)^p$

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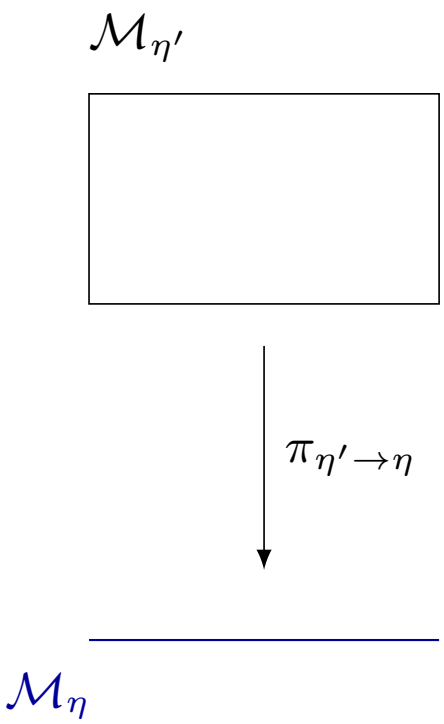
Projective Formalism for QFT

Coarse-graining between Phase Spaces



$\eta \preccurlyeq \eta' \in \mathcal{L}$

Collection of partial theories:

- $\eta \in \mathcal{L}$ = a selection of d.o.f.'s
- 'small' **phase** space $\mathcal{M}_\eta$

Coarse graining:

- projections $\pi_{\eta' \to \eta}$ for $\eta \preccurlyeq \eta'$
- lifting observables
  $A_\eta \circ \pi_{\eta' \to \eta} =: A_{\eta'}$
  ↳ compatible with the Poisson brackets
- **preferred** factorization
  $\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta' \to \eta} \times \mathcal{M}_\eta$

[Projective state spaces: Kijowski '76, Okołów '09 & '13]

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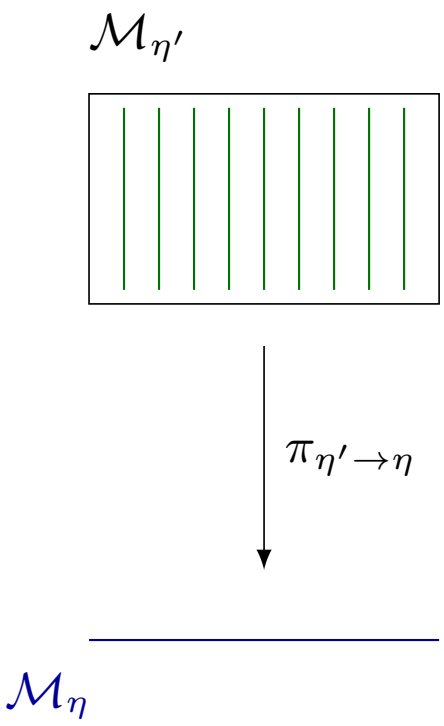

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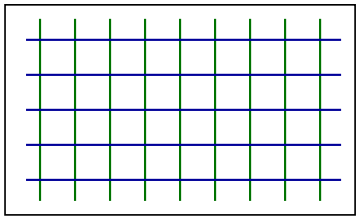

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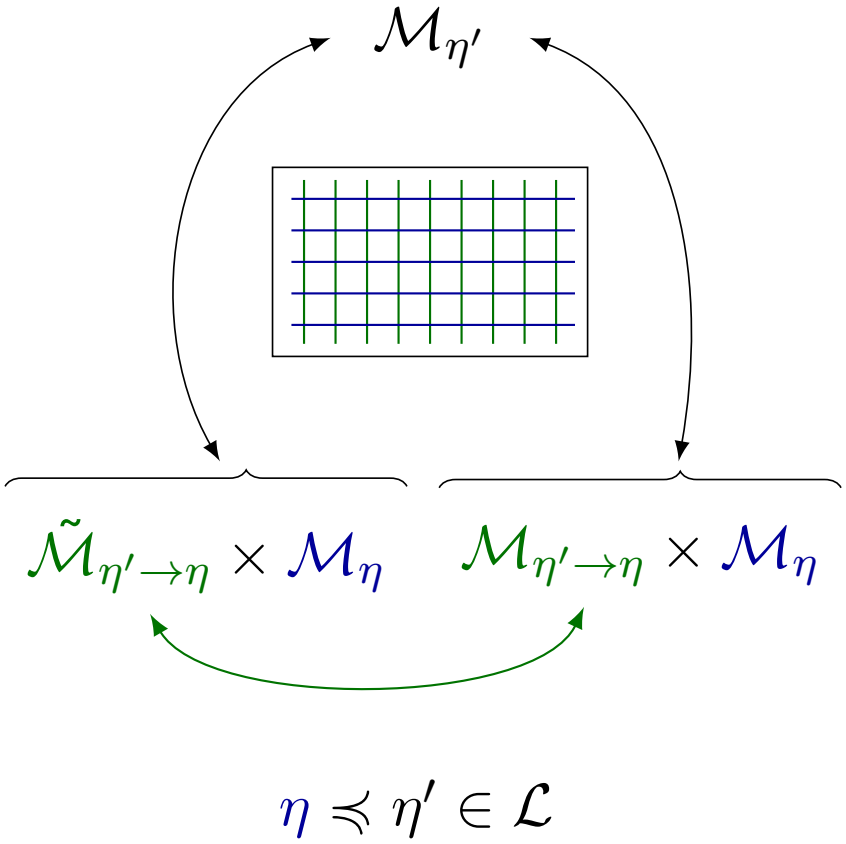

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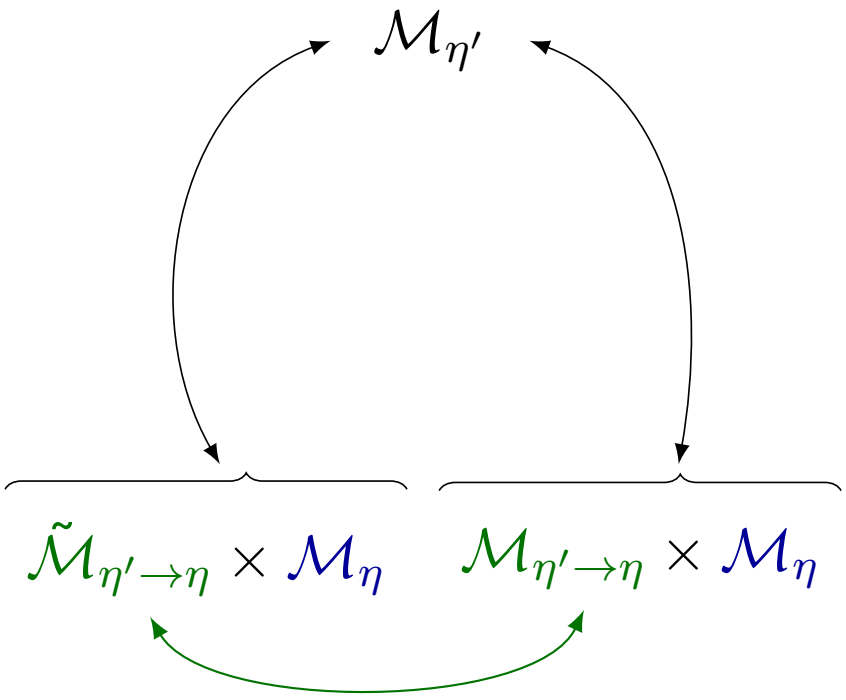

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Quantization:

- classical factorizations
  $\rightsquigarrow$ $\otimes$-factorizations
  $\mathcal{H}_{\eta'} \approx \mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta$
- **unambiguous** $\tilde{\mathcal{H}}_{\eta'\to\eta} \approx \mathcal{H}_{\eta'\to\eta}$

Projective families $(\rho_\eta)_{\eta\in\mathcal{L}}$:

- ρ_η density matrix on $\mathcal{H}_\eta$
- $\rho_\eta = \mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}} \rho_{\eta'} = \mathrm{Tr}_{\tilde{\mathcal{H}}_{\eta'\to\eta}} \rho_{\eta'}$

Ex: linear QFT
$\rightsquigarrow$ **polarization-independant** quantum state space

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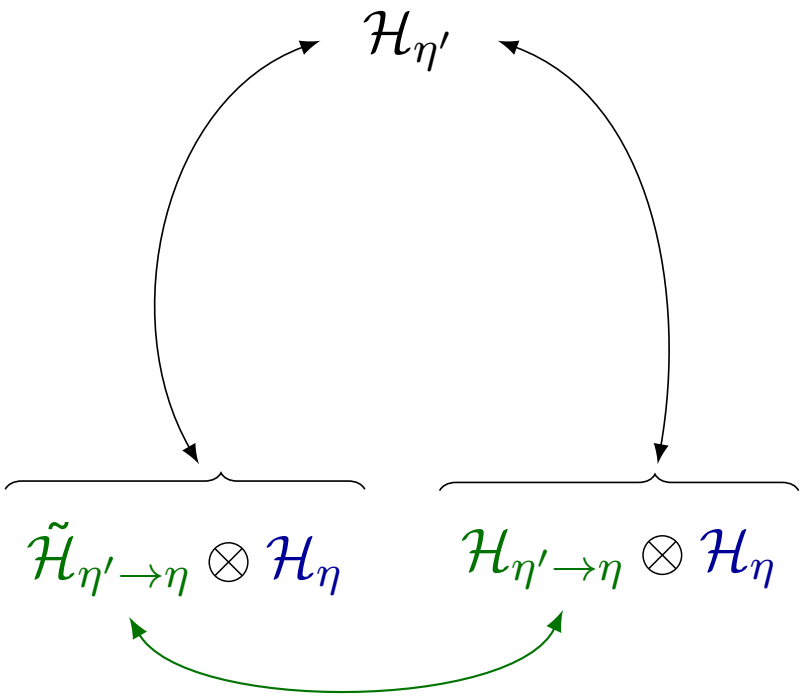

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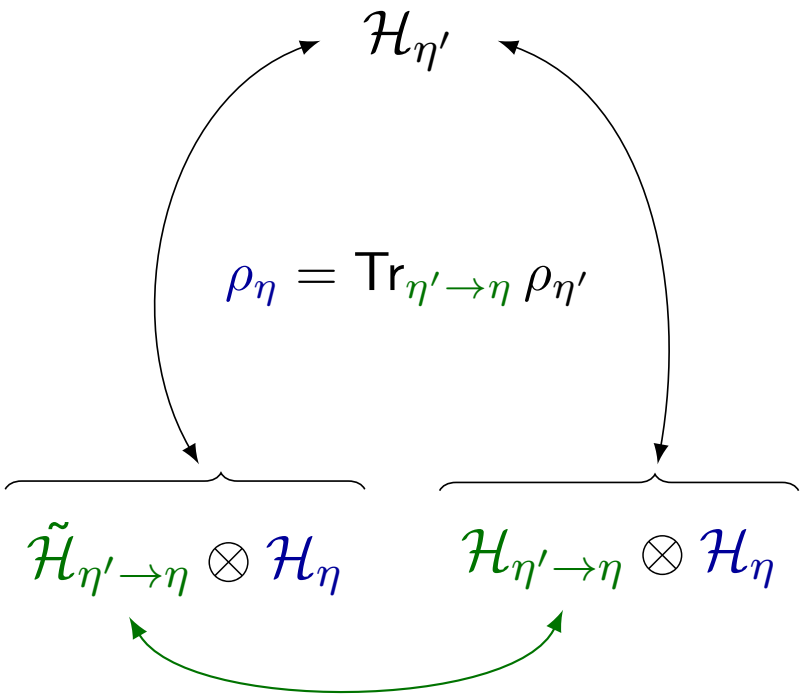

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Projective Formalism for QFT

Natural Embedding of Arbitrary Vacuum Sectors

$$\mathcal{H}_{\eta'} \approx \mathcal{H}_{\eta' \to \eta} \otimes \mathcal{H}_{\eta}$$

$$\rho_{\eta'}$$

$$\Big\downarrow \text{partial trace}$$

$$\mathcal{H}_{\eta}$$

$$\rho_{\eta} = \mathrm{Tr}_{\eta' \to \eta}\, \rho_{\eta'}$$

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Inductive limit $\mathcal{F}_{\varnothing}$:

- choice of **vacuum**
  $\forall \eta \preccurlyeq \eta',\ \left|\varnothing_{\eta' \to \eta}\right\rangle \in \mathcal{H}_{\eta' \to \eta}$
  $\rightsquigarrow$ injection $\iota_{\eta' \leftarrow \eta} : \mathcal{H}_{\eta} \to \mathcal{H}_{\eta'}$
- ex: Fock representation, LQG state space,...

Mapping a density matrix σ on $\mathcal{F}_{\varnothing}$ to a projective state $\left(\rho_{\eta}\right)_{\eta}$:

- this mapping is **injective**
- its image can be characterized

[GNS Representation: Gelfand, Naimark, Segal]

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Natural Embedding of Arbitrary Vacuum Sectors

$$\mathcal{H}_{\eta'} \approx \mathcal{H}_{\eta' \to \eta} \otimes \mathcal{H}_{\eta}$$

$$\left|\varnothing_{\eta' \to \eta}\right\rangle \otimes \left|\psi_{\eta}\right\rangle$$

$$\uparrow \iota_{\eta' \leftarrow \eta}$$

$$\mathcal{H}_{\eta}$$

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- Restriction to a Quasi-Cofinal Sequence of Labels
- Universality & Symmetries

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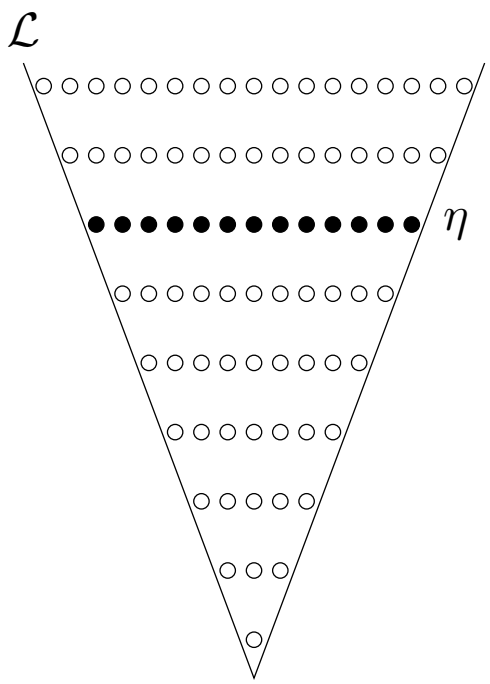
Universal Selections of Degrees of Freedom

Restriction to a Cofinal Sequence of Labels



Sequence $\left(\kappa_n\right)_{n\geqslant 1}$:

- increasing: $\kappa_n \preccurlyeq \kappa_{n+1}$
- **cofinal**: $\forall \eta \in \mathcal{L}, \exists n \,/\, \eta \preccurlyeq \kappa_n$

Bijective mapping $\left(\rho_{\kappa_n}\right)_n \leftrightarrow \left(\rho_\eta\right)_{\eta\in\mathcal{L}}$:

- $\rho_\eta = \mathrm{T}r_{\kappa_n\to\eta}\rho_{\kappa_n}$ for some $\kappa_n \succcurlyeq \eta$
- projective quantum state spaces are **robust** (by contrast eg. to infinite tensor products)

Universal Selections of Degrees of Freedom

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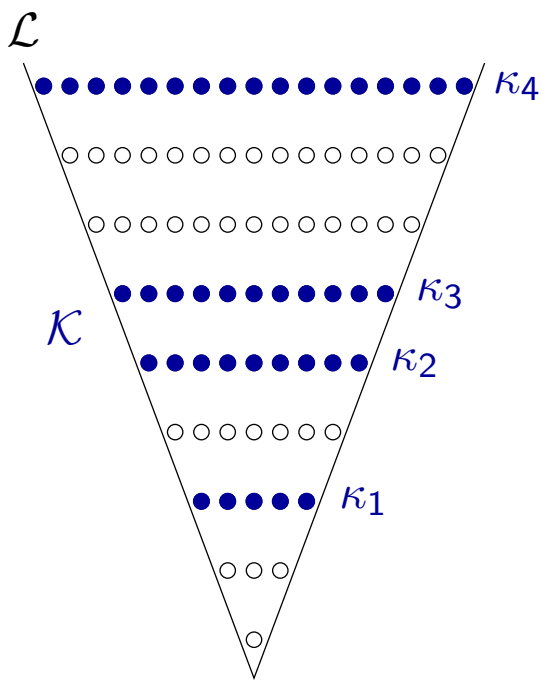

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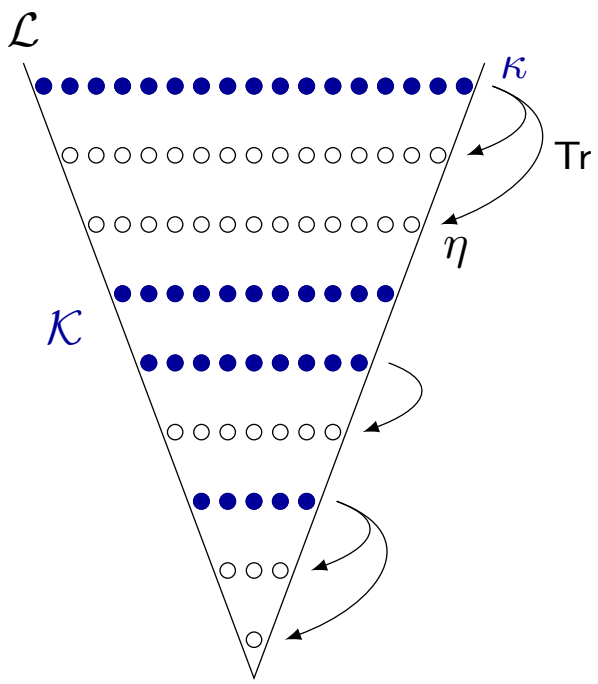

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Universal Selections of Degrees of Freedom

(Semi-classical) States on a Sequence of Labels

$$\mathcal{H}_{\kappa_1}$$

$$\rho_{\kappa_1} = \left|\zeta_{\kappa_1}\right\rangle\!\left\langle\zeta_{\kappa_1}\right|$$

$$\kappa_1 \in \mathcal{L}$$

Semi-classical states on an increasing sequence $\left(\kappa_n\right)_{n\geqslant 1}$:

- first at **macroscopic** scale
- then, step by step, on complementary d.o.f.'s $\kappa_{n+1} \to \kappa_n$ $\rightsquigarrow$ saturating uncertainty relations, given already chosen macro scale behavior

More generally: systematic, **recursive** construction of all projective states $\left(\rho_{\kappa_n}\right)_{n\geqslant 1}$

[See also: Giesel & Thiemann '06; Oriti, Pereira & Sindoni '12]

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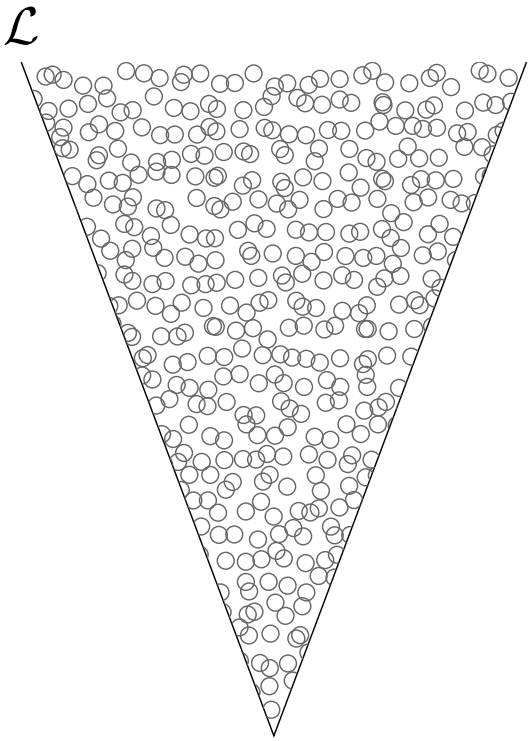

Continuum of d.o.f.'s:

- abstraction for math convenience
- **dense**, **countable** sub-algebra of observables is enough $\rightsquigarrow$ **but:** ensure **universality** and respect **symmetries** of the theory

Quasi-cofinal increasing sequence:
$\forall \eta \in \mathcal{L}, \exists n \geqslant 1 \,/$

- $\tau \cdot \eta \preccurlyeq \kappa_n$ (with τ small deformation)
- $\forall \kappa \preccurlyeq \eta, \kappa_m, \quad \tau \cdot \kappa \overset{!}{=} \kappa$
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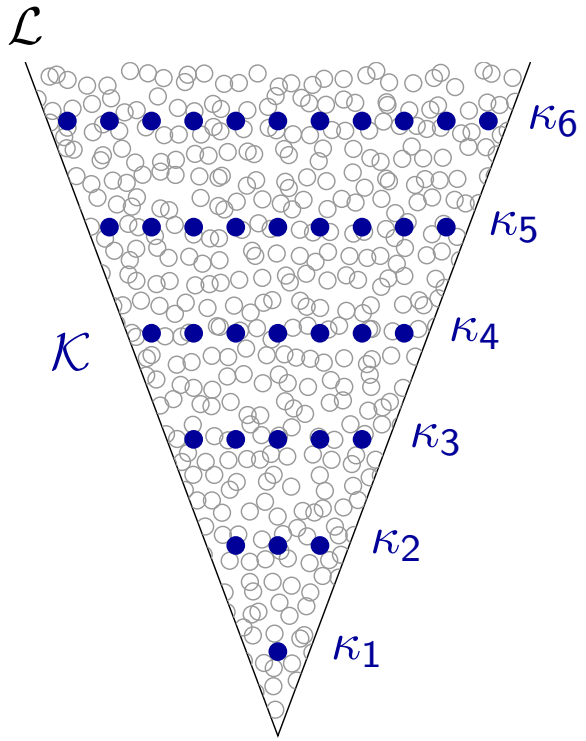

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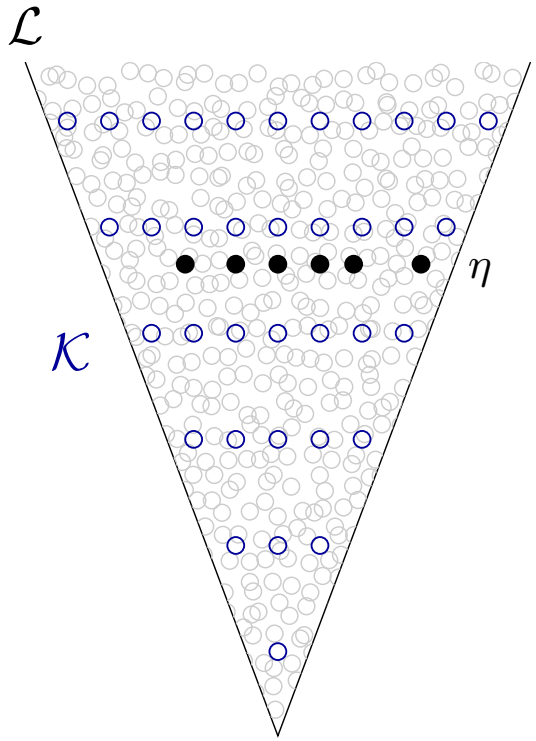

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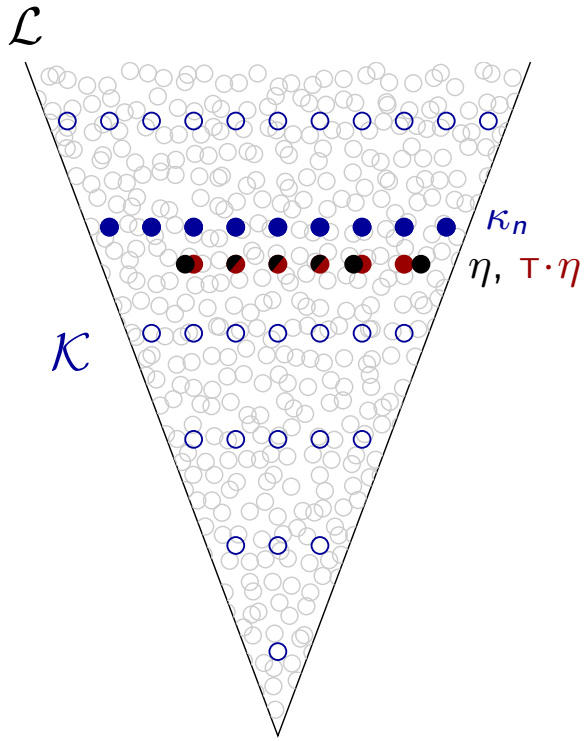

Continuum of d.o.f.'s:

- abstraction for math convenience
- **dense**, **countable** sub-algebra of observables is enough $\rightsquigarrow$ **but:** ensure **universality** and respect **symmetries** of the theory

Quasi-cofinal increasing sequence:
$\forall \eta \in \mathcal{L}, \exists n \geqslant 1 \,/$

- $\tau\cdot\eta \preccurlyeq \kappa_n$ (with τ small deformation)
- $\forall \kappa \preccurlyeq \eta, \kappa_m, \quad \tau\cdot\kappa \overset{!}{=} \kappa$
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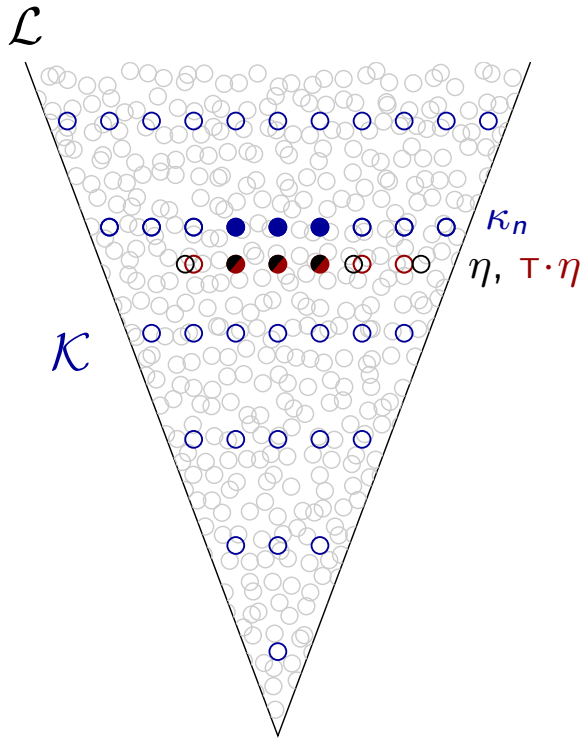

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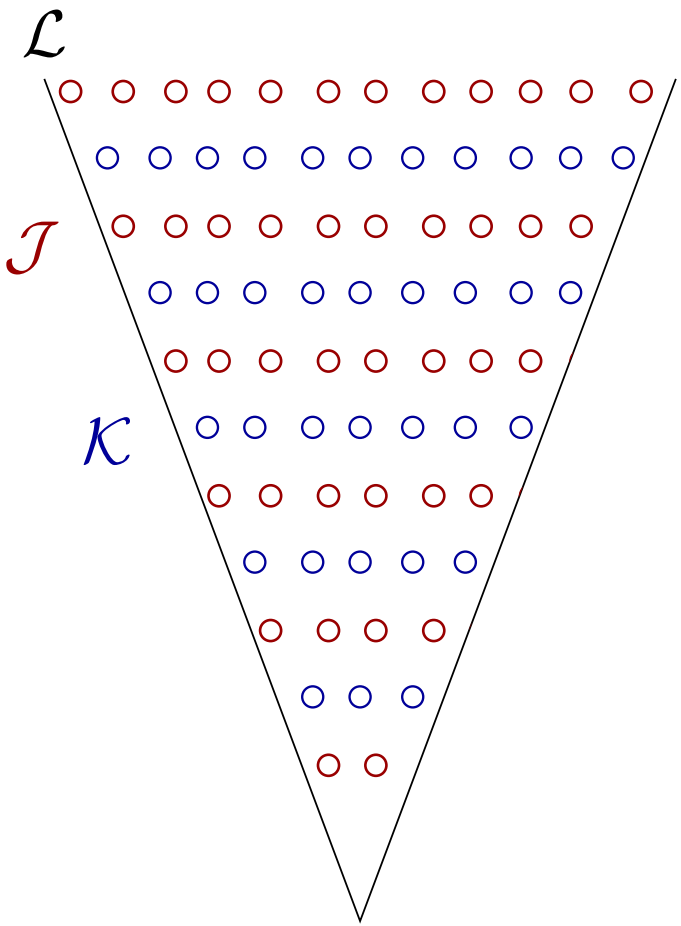
Universality & Symmetries



Subsequences + deformations /

- $\mathsf{s}\cdot\eta_{m_k} \preccurlyeq \mathsf{T}\cdot\kappa_{n_k}$
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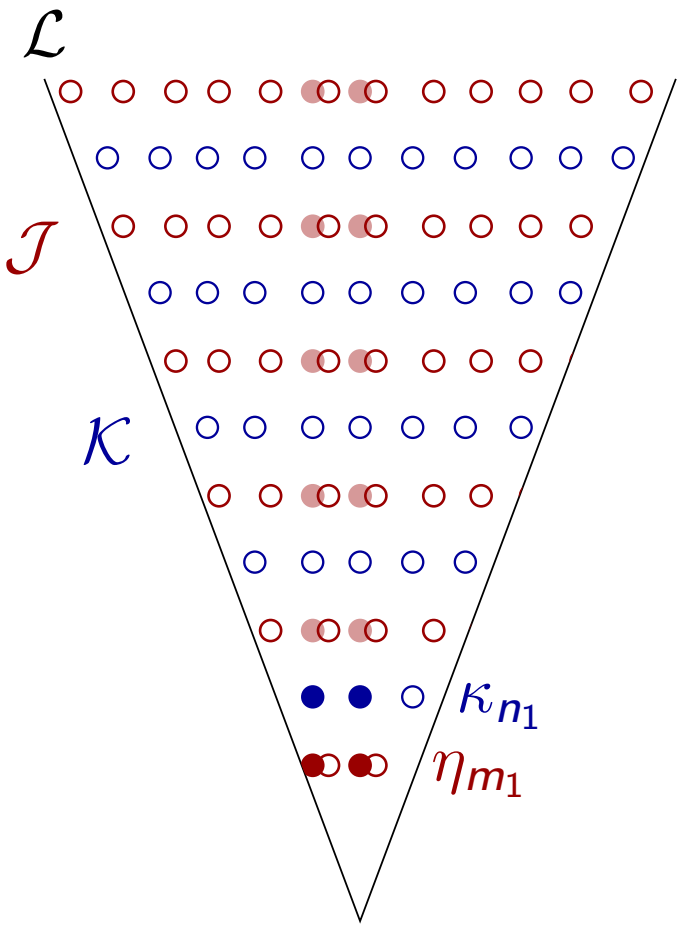

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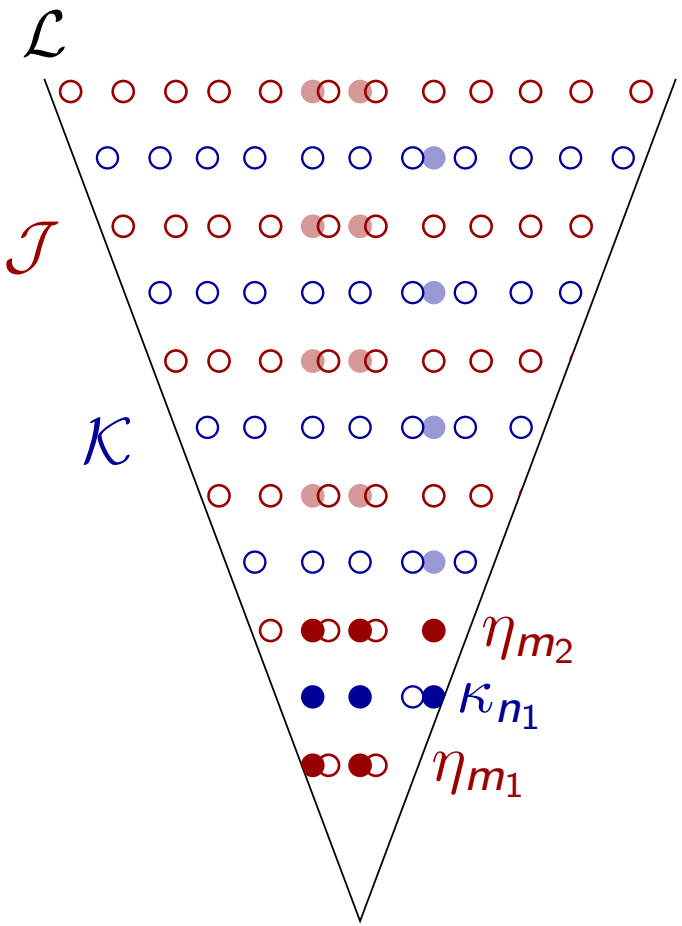

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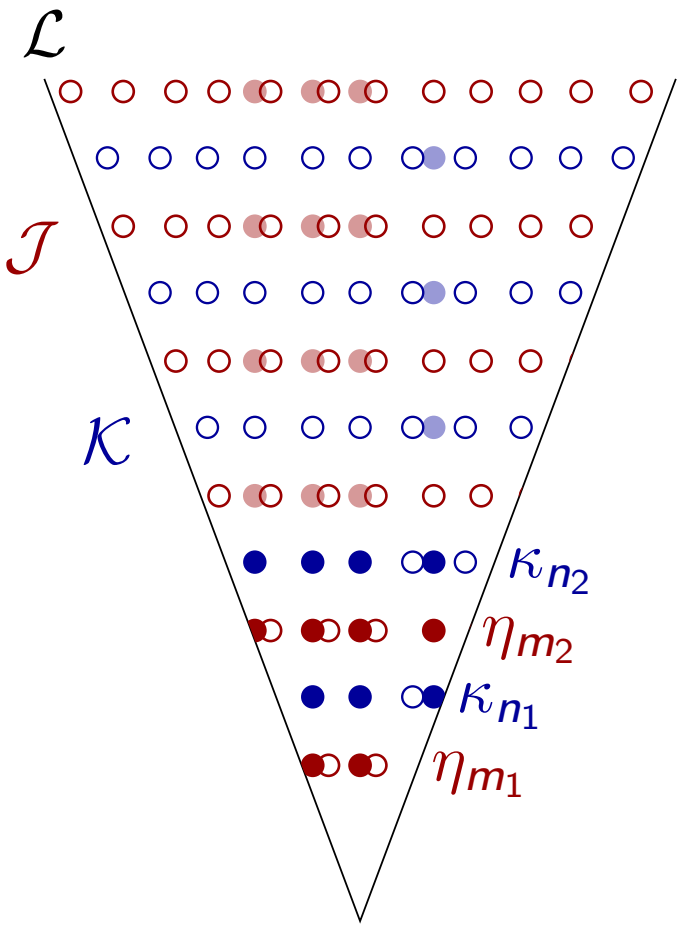

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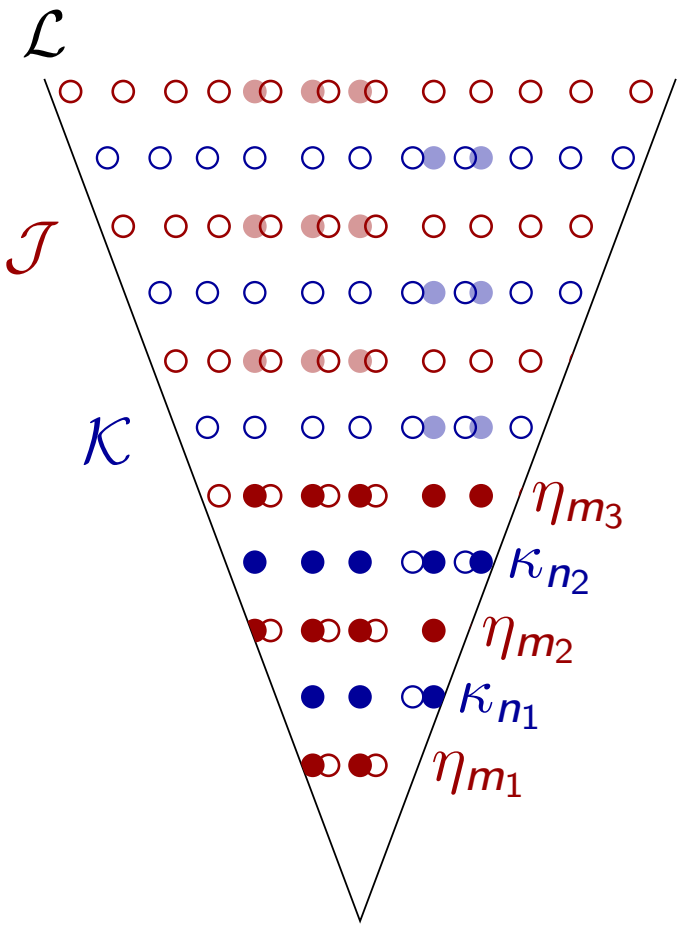

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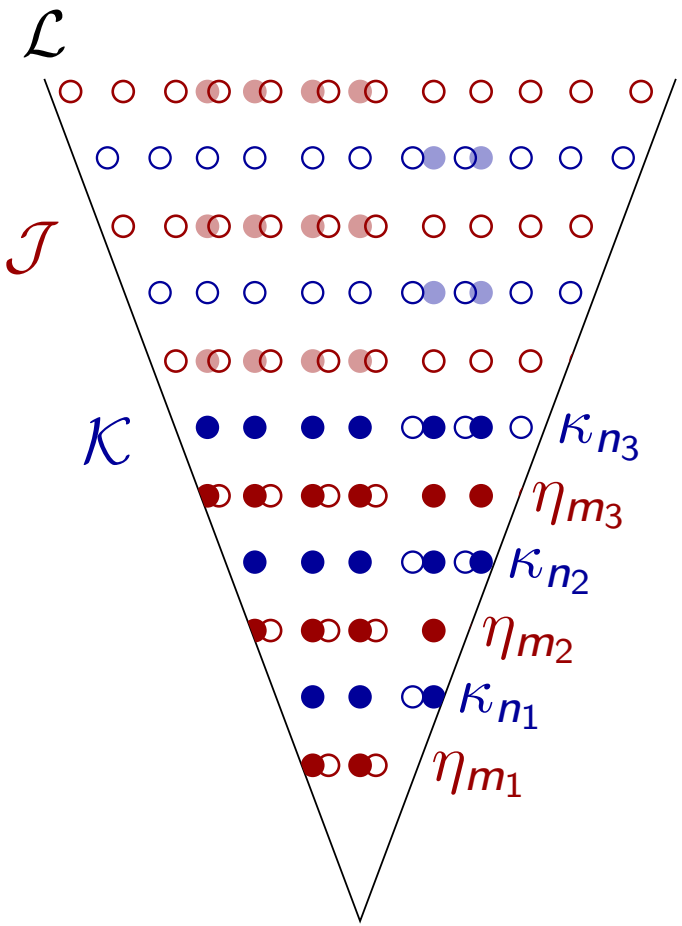

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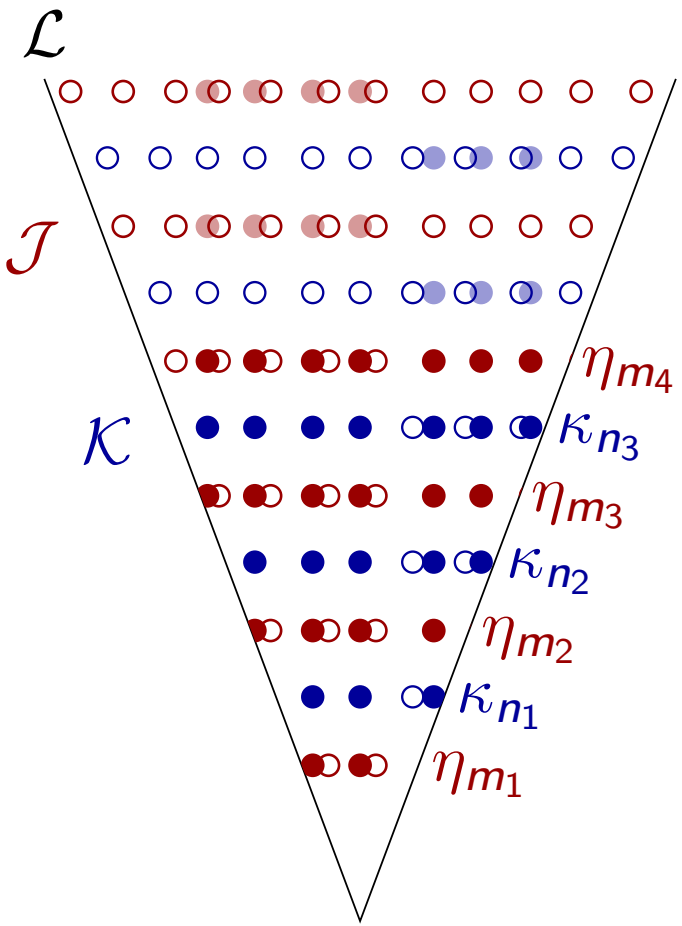

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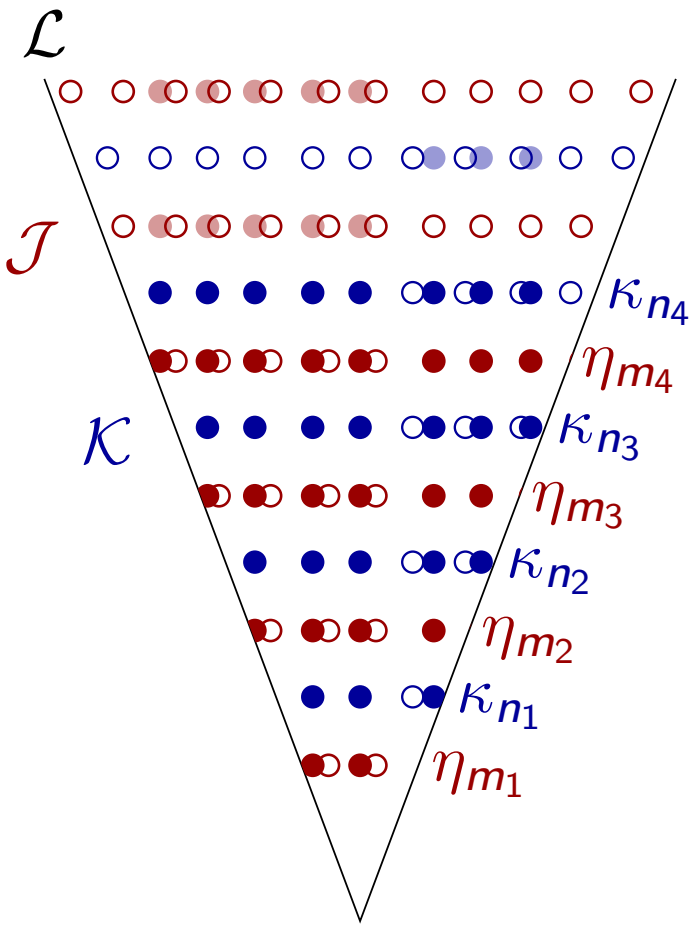

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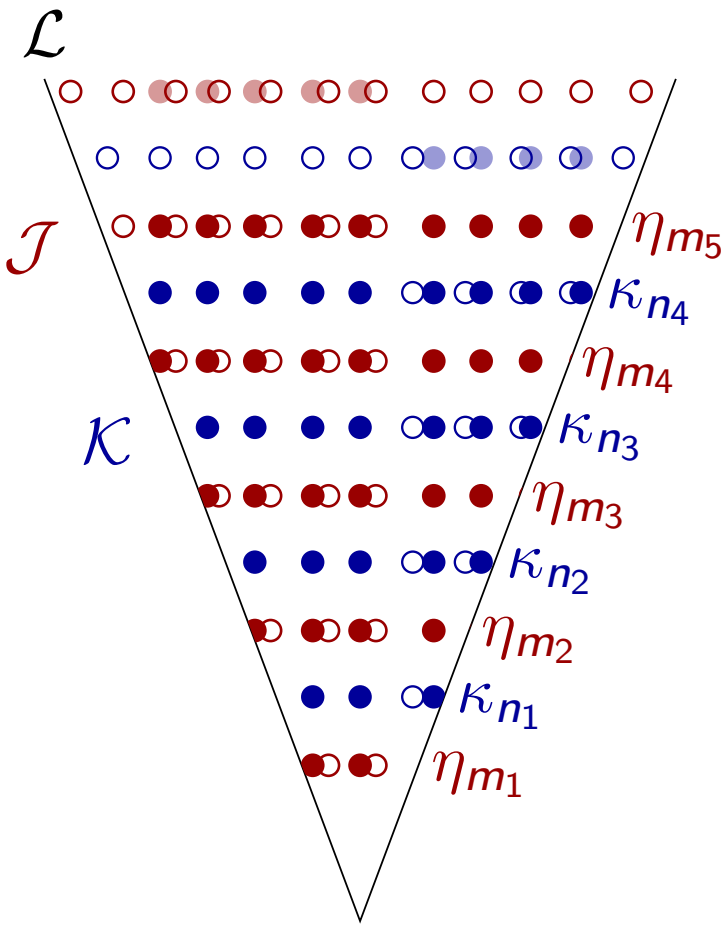

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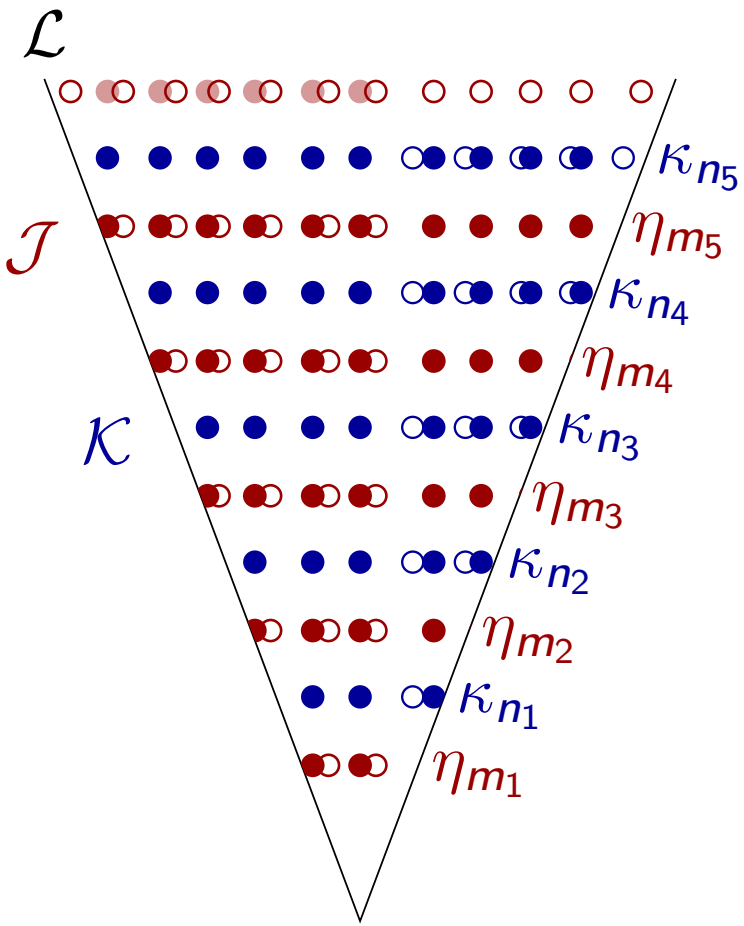

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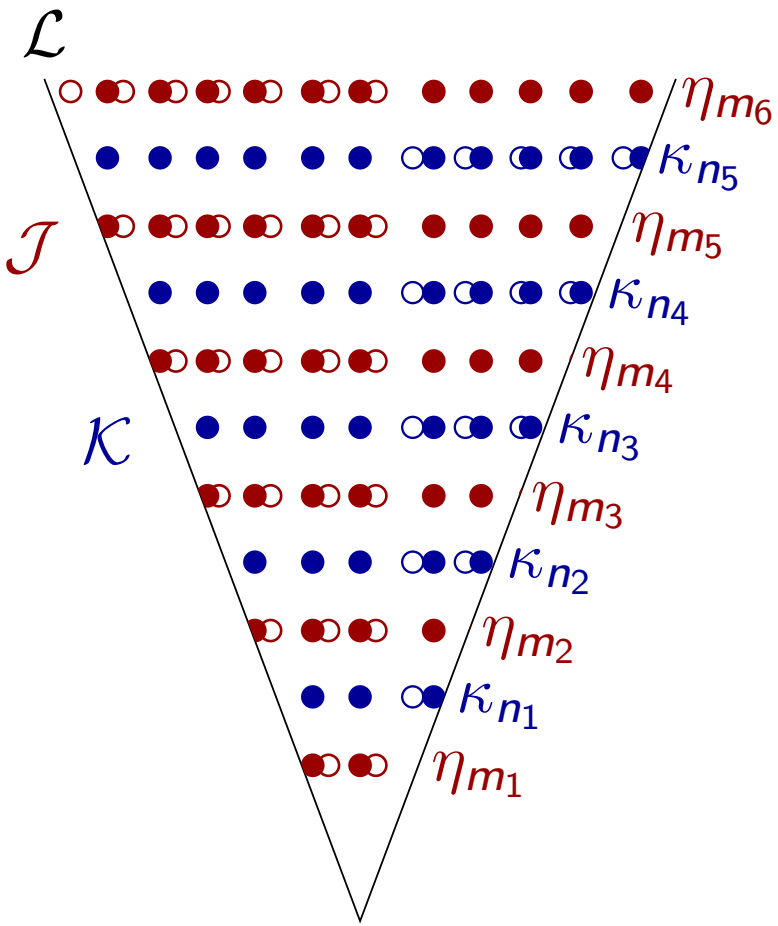

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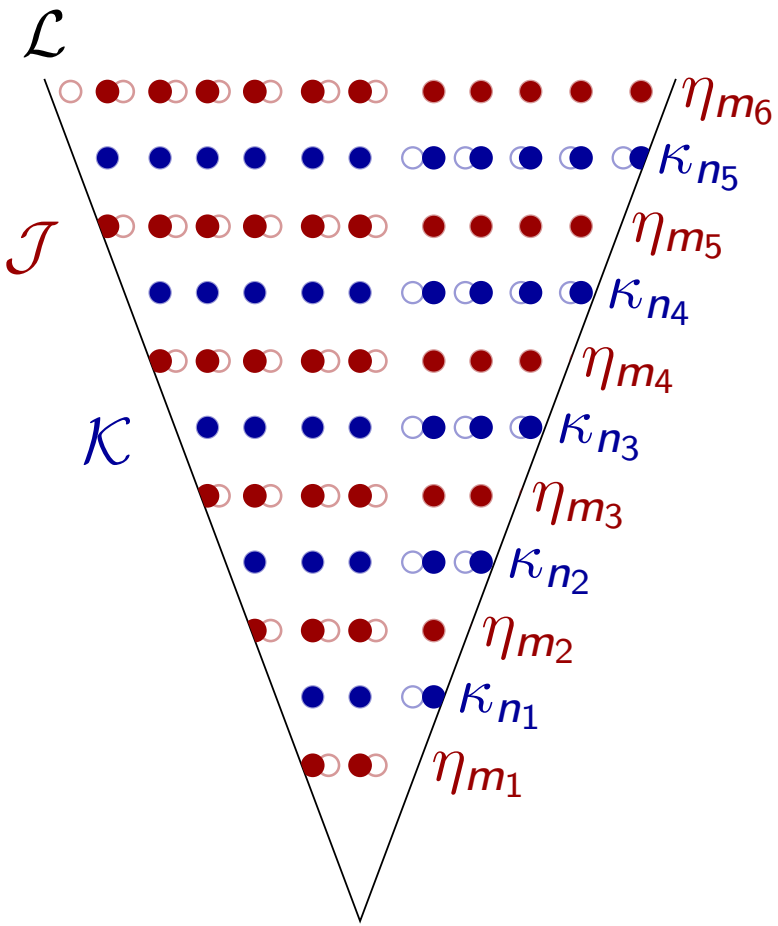

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Projective Formalism for QFT

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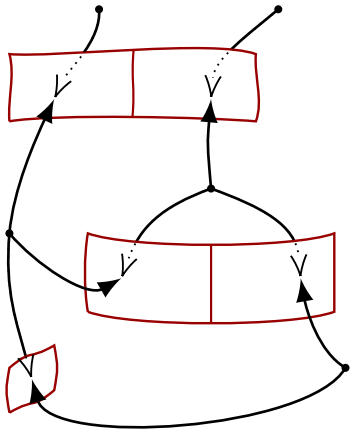
- Relation to the AL Hilbert Space
- Fractal Sequences of Labels

Application to Loop Quantum Gravity

Projective State Space for LQG



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- a graph = holonomies
- a set of surfaces dual to this graph = conjugate fluxes
- $\mathcal{H}_\eta := L_2\left(G^n,\, d\mu_{\text{Haar}}\right)$

The coarse-grainings:

- $G^n \approx G^m \times G^{n-m}$
- kept holonomies $\rightsquigarrow$ prescribes the factor G^m
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[Holonomy-flux algebra: Ashtekar, Gambini, Isham, Lewandowski, Pullin, Rovelli, Smolin,...]

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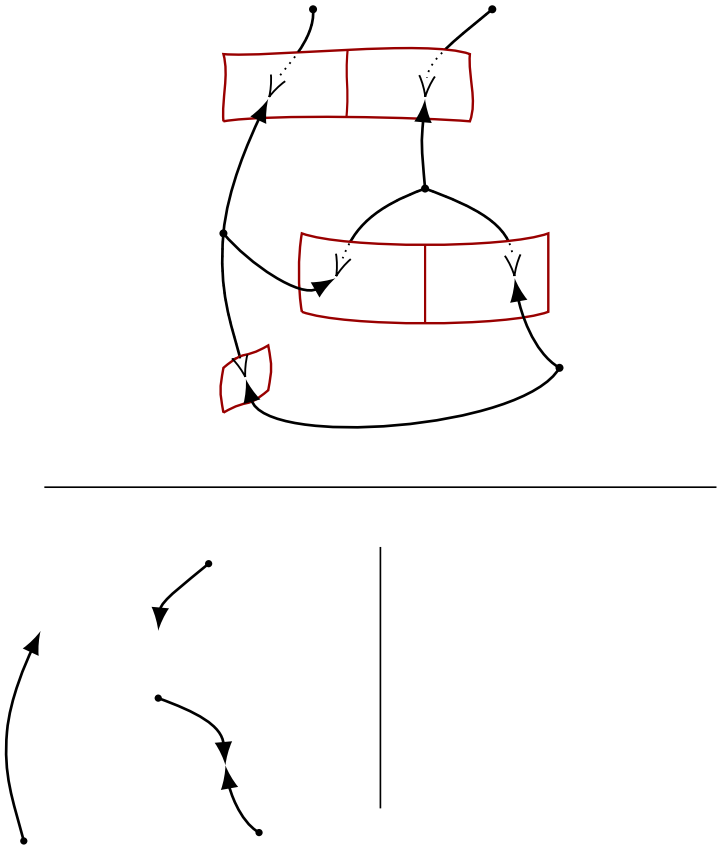

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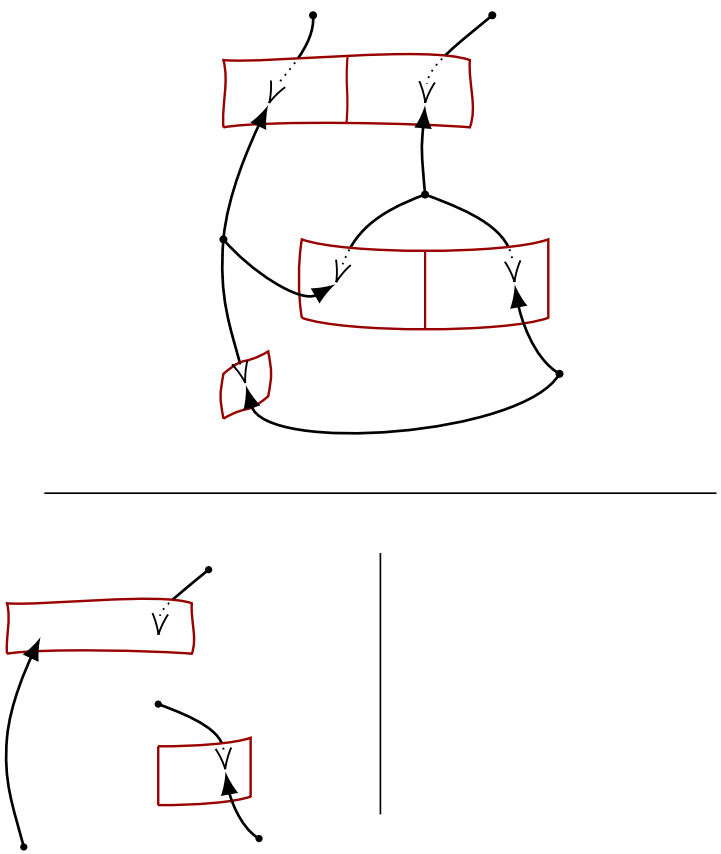

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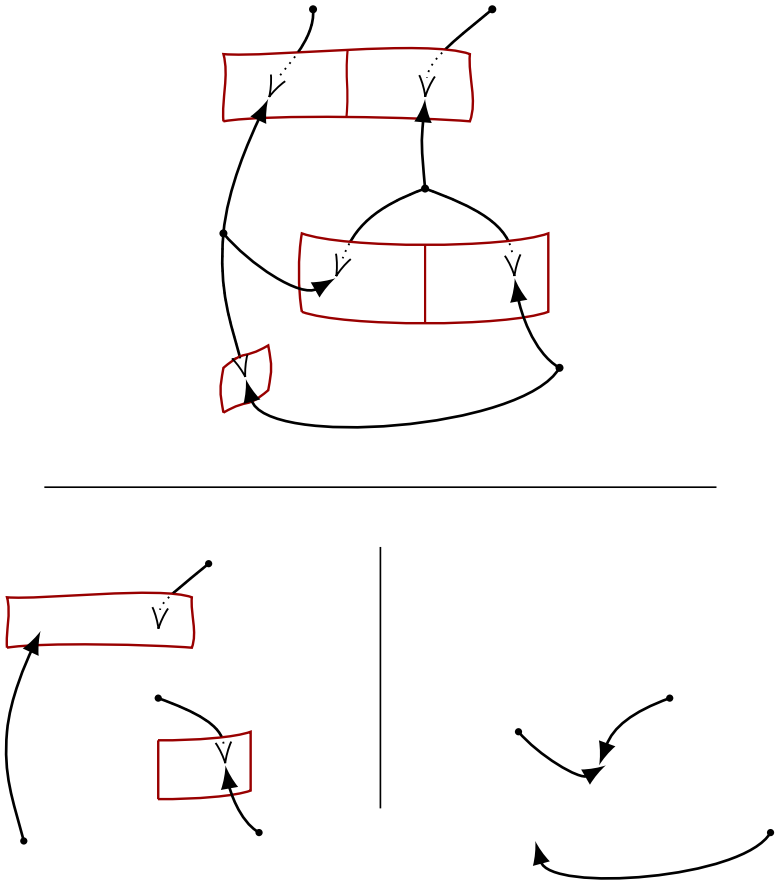

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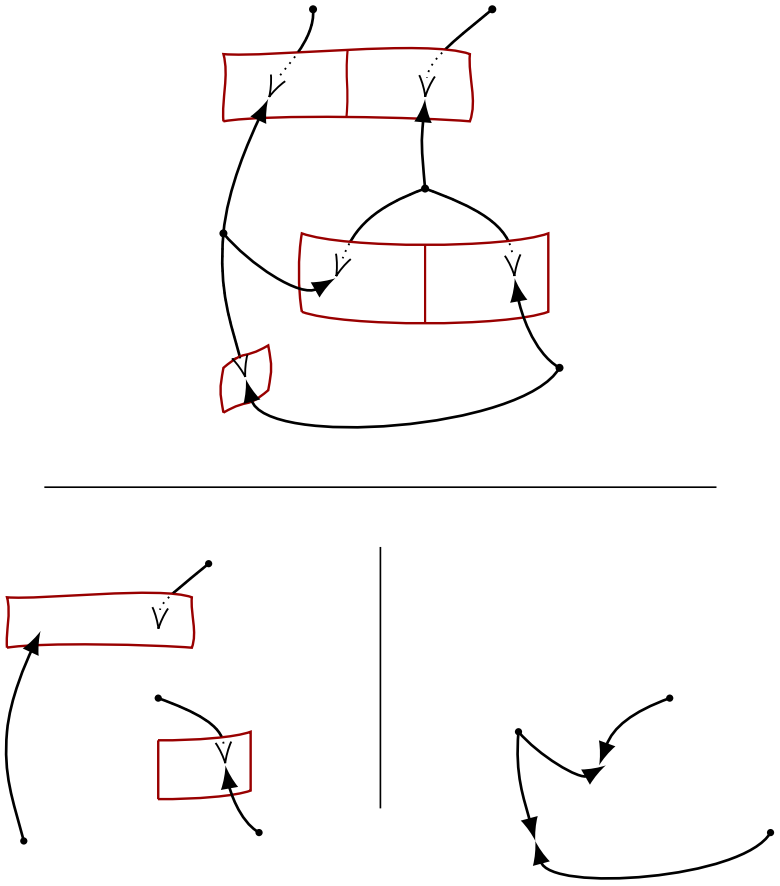

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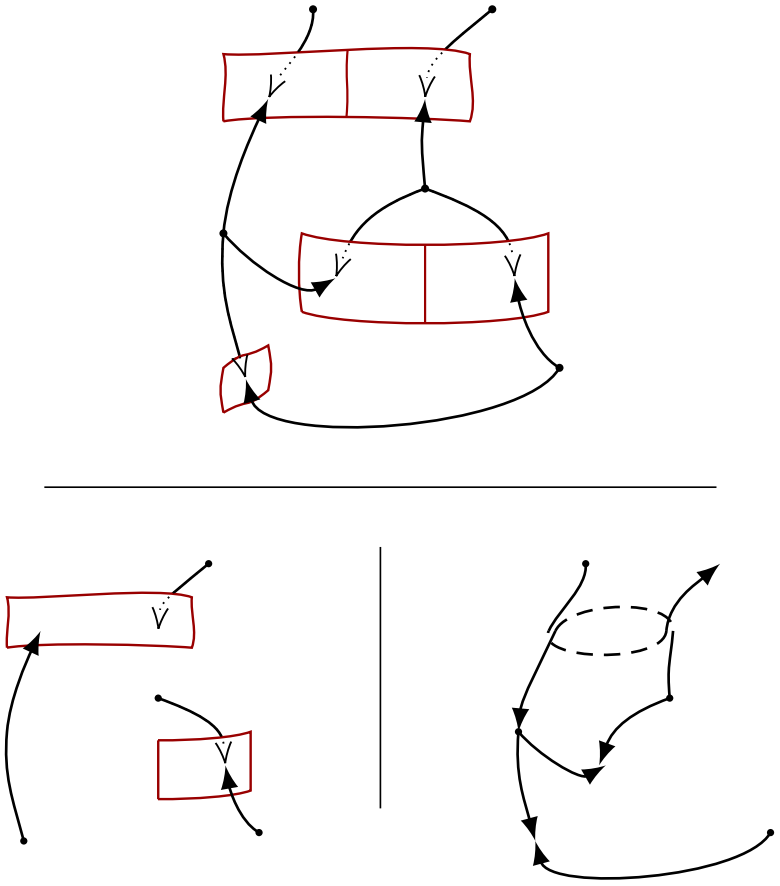

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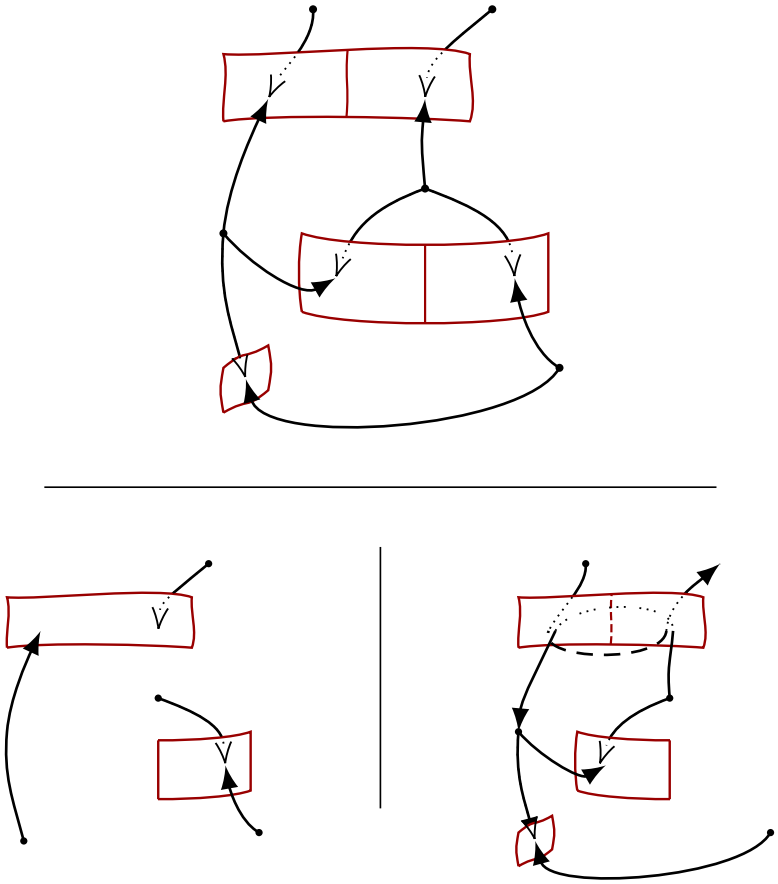
- kept fluxes $\rightsquigarrow$ complementary factor G^{n-m}

[Holonomy-flux algebra: Ashtekar, Gambini, Isham, Lewandowski, Pullin, Rovelli, Smolin,...]

Application to Loop Quantum Gravity

Projective State Space for LQG



The labels:

- a graph = holonomies
- a set of surfaces dual to this graph = conjugate fluxes
- $\mathcal{H}_\eta := L_2\left(G^n,\, d\mu_{\text{Haar}}\right)$

The coarse-grainings:

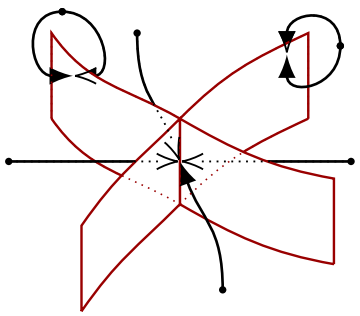
- $G^n \approx G^m \times G^{n-m}$
- kept holonomies $\rightsquigarrow$ prescribes the factor G^m
- kept fluxes $\rightsquigarrow$ complementary factor G^{n-m}

[Holonomy-flux algebra: Ashtekar, Gambini, Isham, Lewandowski, Pullin, Rovelli, Smolin,...]

Application to Loop Quantum Gravity

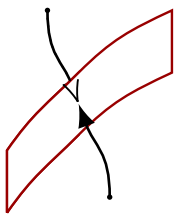
Relation to the AL Hilbert Space

η''



$\psi_\eta(h_1,\, h_2) = \psi_{\eta''}(h_1,\, h_2,\, \ldots)$

η



$\psi_\eta(h_1,\, h_2)$

Inductive limit $\mathcal{H}_{\mathrm{AL}}$:

- AL vacuum: $\varnothing_{\eta' \to \eta} \equiv 1$
  $\in \mathcal{H}_{\eta' \to \eta} = L_2\left(G^{n-m}\right)$
- injection $\mathcal{H}_\eta \to \mathcal{H}_{\eta'}$ only depends on the **graphs**

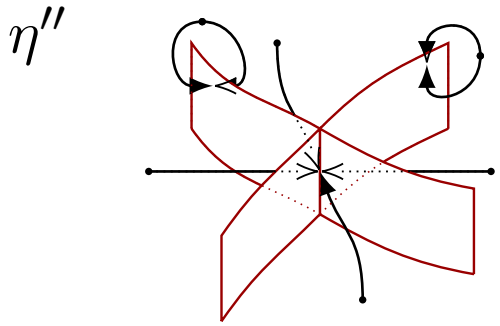
Projective state space extends AL state space:

- holonomies and fluxes on the same footing
- first step toward better semi-classical states

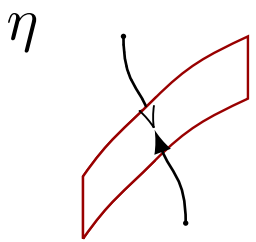
[LQG Hilbert space: Ashtekar, Isham, Lewandowski,...]

Application to Loop Quantum Gravity

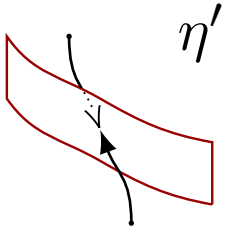
Relation to the AL Hilbert Space



$$\psi_\eta(h_1,\, h_2) = \psi_{\eta''}(h_1,\, h_2,\, \ldots) = \psi_{\eta'}(h_1,\, h_2)$$



$\psi_\eta(h_1,\, h_2)$



$\psi_{\eta'}(h_1,\, h_2)$

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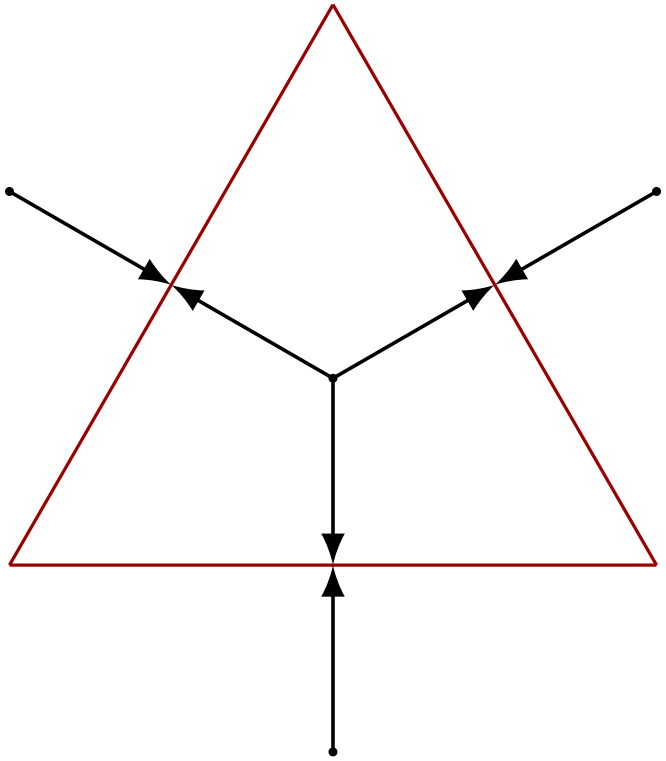
[LQG Hilbert space: Ashtekar, Isham, Lewandowski,...]

Application to Loop Quantum Gravity

Fractal Sequences of Labels



κ_1

Fractal-like sequence of labels:

- **universality** (equivalent state spaces from $\neq$ sequences)
- **background independence** (approx any diffeomorphism by fractal-stabilizing morphism)

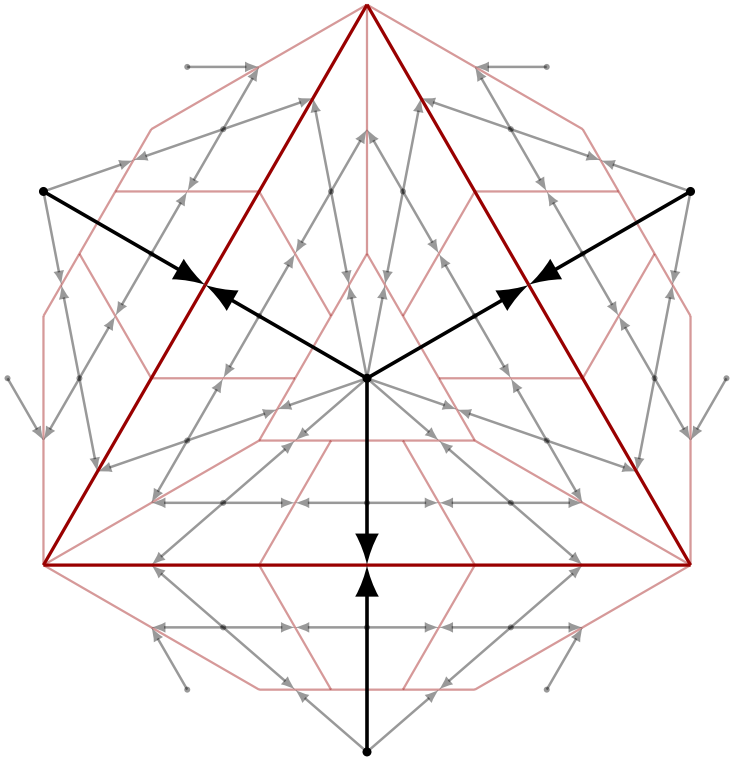
Viz. semi-analytic structure:

- $\neq$ semi-analytic structures related by small deformation
- any diffeo can be approx by semi-analytic one

[See also: Renteln & Smolin, Gambini & Pullin, Corichi & Zapata, Loll,...]

Application to Loop Quantum Gravity

Fractal Sequences of Labels



$\kappa_1 \preccurlyeq \kappa_2$

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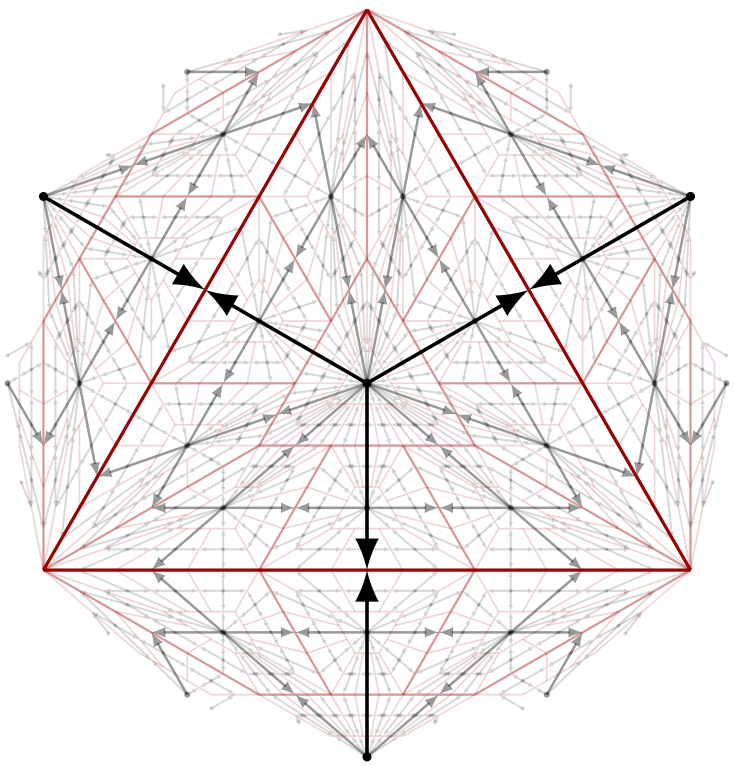

$\kappa_1 \preccurlyeq \kappa_2 \preccurlyeq \kappa_3$

Fractal-like sequence of labels:

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- any diffeo can be approx by semi-analytic one

[See also: Renteln & Smolin, Gambini & Pullin, Corichi & Zapata, Loll,...]

Summary: Comparison with other State Spaces

	analytic holo. & fluxes		infinite graph	fractal-based sub-algebra		
	$\mathcal{H}_{\mathrm{AL}}$	proj. st. space	AQG	induc. limit	proj. st. space	ITP
Hilbert space	✓	✗	✓	✓	✗	✓
Separability	✗	"✗"	✗	✓	"✓"	✗
Universality + Action of diffeos.	✓	✓	✗	✓	✓	✗
Solving Gauss & diffeo. constraints	✓	✗	✓	✓	?	✓
Non-compact G	✗	?	✓	✗	✓	✓
Untruncated semi-cl. states	✗	✗	✓	✗	✓	✓

[ITP: von Neumann '39; AQG: Giesel & Thiemann '06,...]

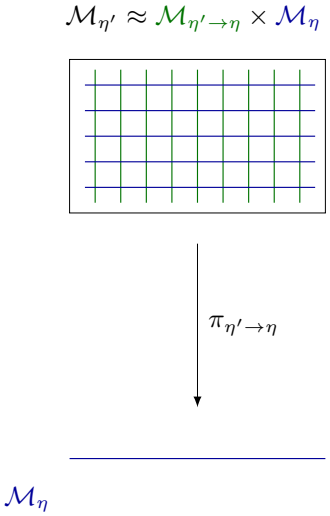
Outlook

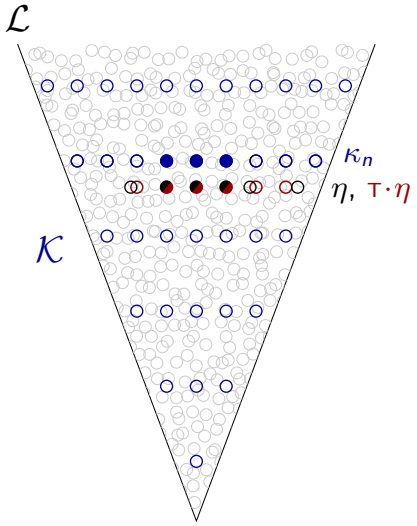
Applications of the thus constructed state space:

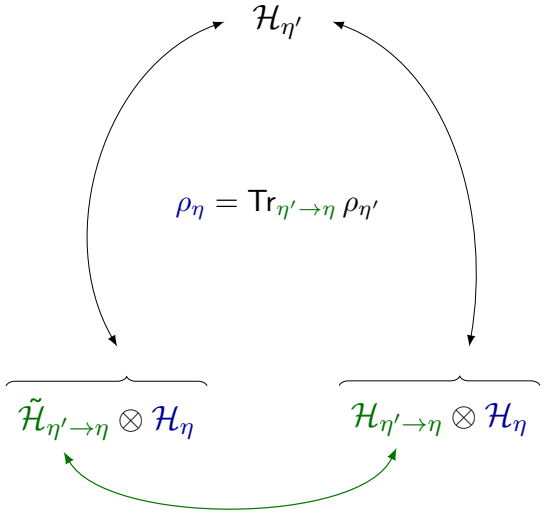
- study of the **semi-classical regime** of loop quantum gravity
- derivation of **symmetry reduced models** (LQC, spherically symmetric LQG,...) from the full theory [Engle '07]

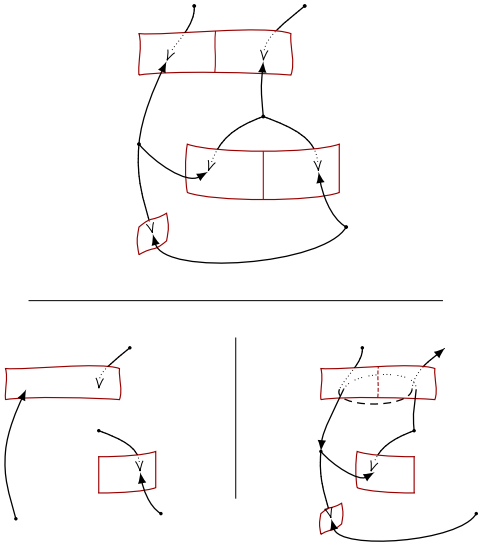
Implementation of the **constraints**:

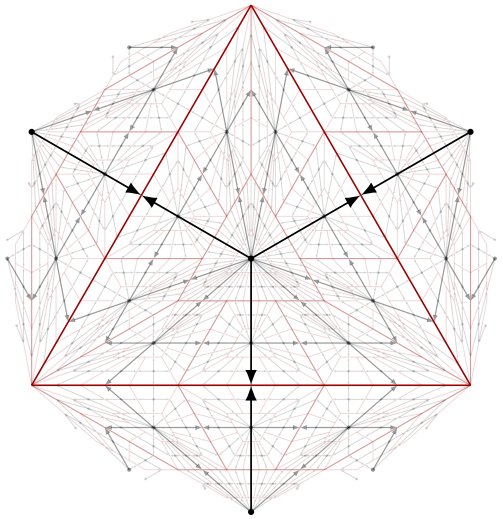
- general prescriptions to deal with constraints in the context of projective state spaces: in particular, relations with **covariant approaches**, renormalization...
- **Gauss** & **diffeomorphism** constraints: somewhat easier when working on fractal sequence of labels
- dynamics...


$\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$
$\pi_{\eta'\to\eta}$
$\mathcal{M}_\eta$


$\mathcal{L}$
κ_n
$\eta,\ \tau\cdot\eta$
$\mathcal{K}$


$\mathcal{H}_{\eta'}$
$\rho_\eta = \mathrm{Tr}_{\eta'\to\eta}\,\rho_{\eta'}$
$\tilde{\mathcal{H}}_{\eta'\to\eta} \otimes \mathcal{H}_\eta$
$\mathcal{H}_{\eta'\to\eta} \otimes \mathcal{H}_\eta$





Thank you!

Contents

Extra Slides
- Projective State Spaces
- Relation to Other Constructions
- Implementation of Gauge Constraints and Dynamics
- Basics of (Loop) Quantum Gravity
- Projective State Space for the Holonomy-Flux Algebra
- Obstruction to the Construction of Semi-classical States

Classical Projective State Spaces

Projective Systems of Phase Spaces

$$\begin{tikzcd}
 & \mathcal{M}_{\eta''} \arrow[dl, bend right, "\pi_{\eta''\to\eta'}"'] \arrow[ddl, bend left, "\pi_{\eta''\to\eta}"] \\
\mathcal{M}_{\eta'} \arrow[d, bend left, "\pi_{\eta'\to\eta}"'] & \\
\mathcal{M}_{\eta} &
\end{tikzcd}$$

$$\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$$

Collection of partial theories:

- label set $\mathcal{L}$, $\preccurlyeq$
- $\eta \in \mathcal{L}$ = a selection of d.o.f.'s
- 'small' phase spaces $\mathcal{M}_\eta$

Ensuring **consistency**:

- projections $\pi_{\eta'\to\eta}$ for $\eta \preccurlyeq \eta'$
- mounting observables:
  $O_{\eta'} = O_\eta \circ \pi_{\eta'\to\eta}$
- 3-spaces-consistency
  $\to$ projective system

[Projective state spaces: Kijowski '76, Okołów '09 & '13]

Projective State Spaces

Projections and Factorizations

$$\pi : \mathcal{M} \to \widetilde{\mathcal{M}}$$

$$\begin{array}{c} q_1, \ \ldots\ldots, \ q_n \\ p_1, \ \ldots\ldots, \ p_n; \\ \downarrow \\ \widetilde{q}_1, \ \ldots, \ \widetilde{q}_m \\ \widetilde{p}_1, \ \ldots, \ \widetilde{p}_m; \end{array}$$

Projective State Spaces

Projections and Factorizations

$$\pi : \mathcal{M} \to \widetilde{\mathcal{M}} \qquad\qquad \mathcal{M} \approx \widetilde{\mathcal{M}} \times \underset{\sim}{\mathcal{M}}$$

$$\begin{array}{c} q_1, \ \ldots\ldots, \ q_n \\ p_1, \ \ldots\ldots, \ p_n; \\ \downarrow \\ \widetilde{q}_1, \ \ldots, \ \widetilde{q}_m \\ \widetilde{p}_1, \ \ldots, \ \widetilde{p}_m; \end{array} \qquad\qquad \begin{array}{c} q_1, \ \ldots\ldots, \ q_n \\ p_1, \ \ldots\ldots, \ p_n; \\ \updownarrow \\ \widetilde{q}_1, \ \ldots, \ \widetilde{q}_m, \ \underset{\sim}{q}_{m+1}, \ \ldots, \ \underset{\sim}{q}_n \\ \widetilde{p}_1, \ \ldots, \ \widetilde{p}_m, \ \underset{\sim}{p}_{m+1}, \ \ldots, \ \underset{\sim}{p}_n; \end{array}$$

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$$\neq$$

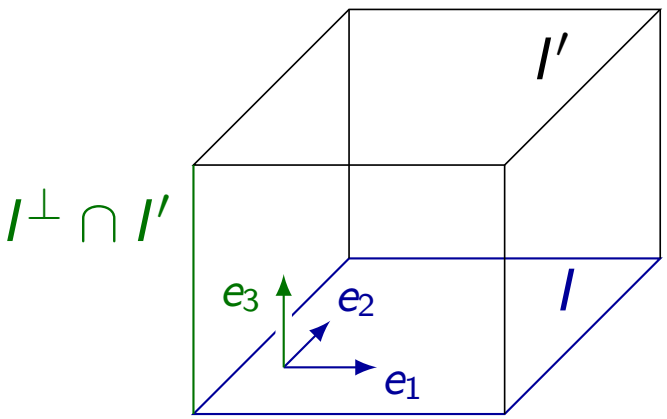
$$\tau : \mathcal{C} \to \widetilde{\mathcal{C}}$$

$$\begin{array}{c} q_1, \ \ldots\ldots, \ q_n \\ \downarrow \\ \widetilde{q}_1, \ \ldots, \ \widetilde{q}_m \end{array}$$

Classical Projective State Spaces

Example: Second Quantization

$$I' \approx \left(I^\perp \cap I'\right) \times I$$



$I \subset I' \subset \mathcal{H}_{\text{one particule}}$

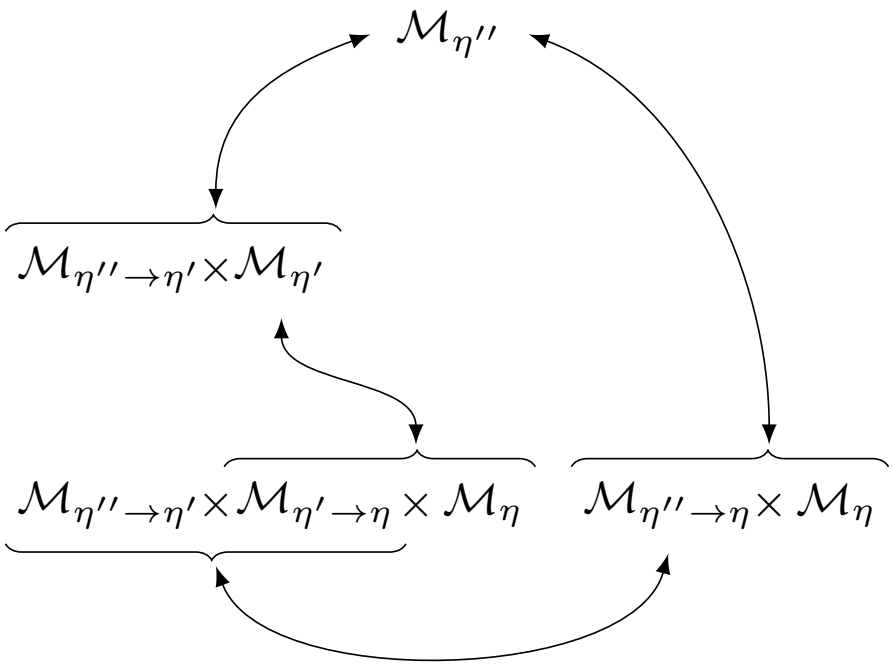
First quantized theory:

- Hilbert space $\mathcal{H}_{\text{one particule}}$
  $\omega_{\text{symplect}} := 2\,\mathrm{Im}\,\langle\cdot,\,\cdot\rangle$
- $\mathsf{a}_e : \psi \mapsto \langle e,\,\psi\rangle$
- $\{\mathsf{a}_e\,,\,\mathsf{a}_f^*\} = i\,\langle e,\,f\rangle$

Projective system:

- labels: $I \subset \mathcal{H}_{\text{one particule}}$
  with $\dim I < \infty$
- 'small' phase space: $\mathcal{M}_I := I$
- $\pi_{I' \to I}$ orthog. projection on $I \subset I'$
- $\exists!$ factorization: $I' \approx \left(I^\perp \cap I'\right) \times I$

Quantum Projective State Spaces



$\eta \preccurlyeq \eta' \preccurlyeq \eta'' \in \mathcal{L}$

Modeled on special case:

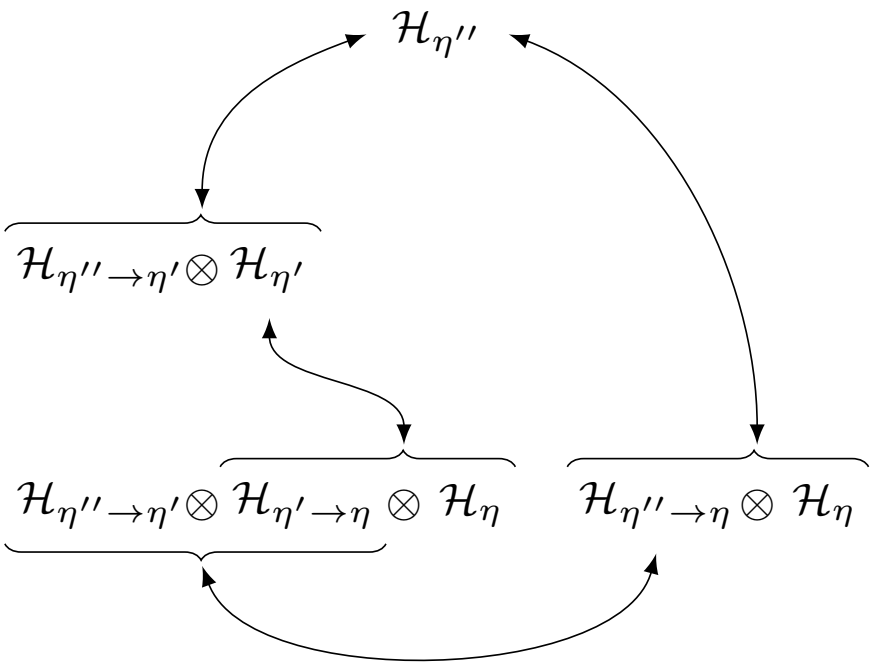
- classical factorizations
  $\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_{\eta}$
- 3-spaces-consistency
  $\mathcal{M}_{\eta''\to\eta} \approx \mathcal{M}_{\eta''\to\eta'} \times \mathcal{M}_{\eta'\to\eta}$
- quantization
  $\rightsquigarrow$ $\bigotimes$-factorizations

Projective families $(\rho_\eta)_{\eta\in\mathcal{L}}$:

- ρ_η density matrix on $\mathcal{H}_\eta$
- $\mathrm{Tr}_{\mathcal{H}_{\eta'\to\eta}}\, \rho_{\eta'} = \rho_\eta$

[Projective state spaces: Kijowski '76, Okołów '09 & '13]

Quantum Projective State Spaces



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[Projective state spaces: Kijowski '76, Okołów '09 & '13]

Projective State Spaces

Quantization

	$\mathcal{M}_\eta \quad \dashrightarrow \quad \mathcal{H}_\eta$	$\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta' \to \eta} \times \mathcal{M}_\eta$ $\dashrightarrow \quad \mathcal{H}_{\eta'} \approx \mathcal{H}_{\eta' \to \eta} \otimes \mathcal{H}_\eta$
Holo. repr.	Holomorphic st. on $\mathcal{M}_\eta$ $\mathcal{H}_\eta = L_2(\mathcal{M}_\eta\,,\, d\upsilon_\eta) \cap \text{Holo}$	$\exists$ holomorphic st. on $\mathcal{M}_{\eta' \to \eta}$ such that $\approx$ is holomorphic
Position repr.	$\mathcal{M}_\eta = T^*(\mathcal{C}_\eta)$ $\mathcal{H}_\eta = L_2(\mathcal{C}_\eta\,,\, d\mu_\eta)$	$\mathcal{C}_{\eta'} \approx \mathcal{C}_{\eta' \to \eta} \times \mathcal{C}_\eta$ $\mu_{\eta'} \approx \mu_{\eta' \to \eta} \times \mu_\eta$

$+$ observables...

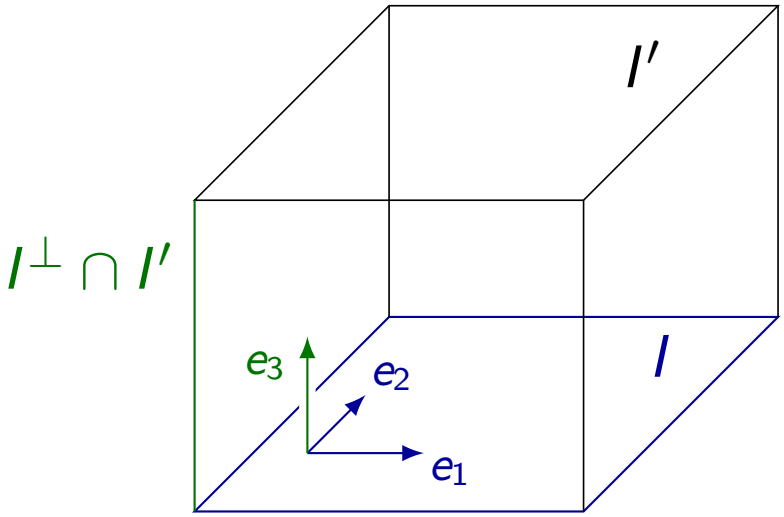
[Geometric quantization: Woodhouse '92]

Quantum Projective State Spaces

Example: Second Quantization

$$\mathcal{F}_{I'} \approx \mathcal{F}_{I^\perp \cap I'} \otimes \mathcal{F}_I$$

$$\left|n_1,\, n_2,\, n_3\right\rangle = \left|n_3\right\rangle \otimes \left|n_1,\, n_2\right\rangle$$



$I \subset I' \subset \mathcal{H}_{\text{one particule}}$

Projective system for the second quantization of $\mathcal{H}_{\text{one particule}}$:

- labels: $I \subset \mathcal{H}_{\text{one particule}}$ with $\dim I < \infty$
- 'small' Hilbert space: $\mathcal{F}_I =$ partial Fock space built on I
- $I \subset I' \Rightarrow \mathcal{F}_{I'} \approx \mathcal{F}_{I^\perp \cap I'} \otimes \mathcal{F}_I$

Mapping a density matrix σ on $\mathcal{F}_{\text{Fock}}$ to a projective state $(\rho_I)_I$:

- $\forall\, I,\ \rho_I = \mathrm{Tr}_{\mathcal{F}_{I^\perp}}\, \sigma$ using $\mathcal{F}_{\text{Fock}} \approx \mathcal{F}_{I^\perp} \otimes \mathcal{F}_I$

Inductive Limit from a Choice of Vacuum

Fock Space

Fock space $\mathcal{F}_{\text{Fock}}$ built on $\mathcal{H}_{\text{one particule}}$ as an inductive limit:

- $\varnothing_{I' \to I} = \left|(0)_j\right\rangle \in \mathcal{F}_{I^\perp \cap I'}$
- $\iota_{I' \leftarrow I} : \left|(n_i)_i\right\rangle \in \mathcal{F}_I \mapsto \left|(0)_j\,,\,(n_i)_i\right\rangle \in \mathcal{F}_{I'} \approx \mathcal{F}_{I^\perp \cap I'} \otimes \mathcal{F}_I$

Mapping a density matrix σ on $\mathcal{F}_{\text{Fock}}$ to a projective state $(\rho_I)_I$:

- $\forall\, I \subset \mathcal{H}_{\text{one particule}},\ \rho_I = \mathrm{Tr}_{\mathcal{F}_{I^\perp}}\, \sigma$
  using the factorization $\mathcal{F}_{\text{Fock}} \approx \mathcal{F}_{I^\perp} \otimes \mathcal{F}_I$
- $\sup\limits_{I}\ \inf\limits_{I' \supset I}\ \mathrm{Tr}_{\mathcal{F}_{I'}} \left[\left(|\varnothing_{I' \to I}\rangle\langle\varnothing_{I' \to I}| \otimes \mathbf{1}_{\mathcal{F}_I} \right)\ \rho_{I'} \right] = 1$

Infinite Tensor Product

Infinite tensor product

$$\bigotimes_{f\in\Lambda} \mathcal{J}_f := \bigoplus_{[\Omega]_\simeq} \mathcal{H}_{[\Omega]_\simeq}$$

$$\Omega \simeq \Omega' \Leftrightarrow \sum_{f\in\Lambda} \left|\langle \Omega_f, \Omega'_f\rangle - 1\right| < \infty$$

Projective system:

- $\lambda \subset \Lambda$, $\#\lambda < \infty$
- $\mathcal{H}_\lambda := \bigotimes_{f\in\lambda} \mathcal{J}_f$

ITP states $\leftarrow\dashrightarrow$ projective states:

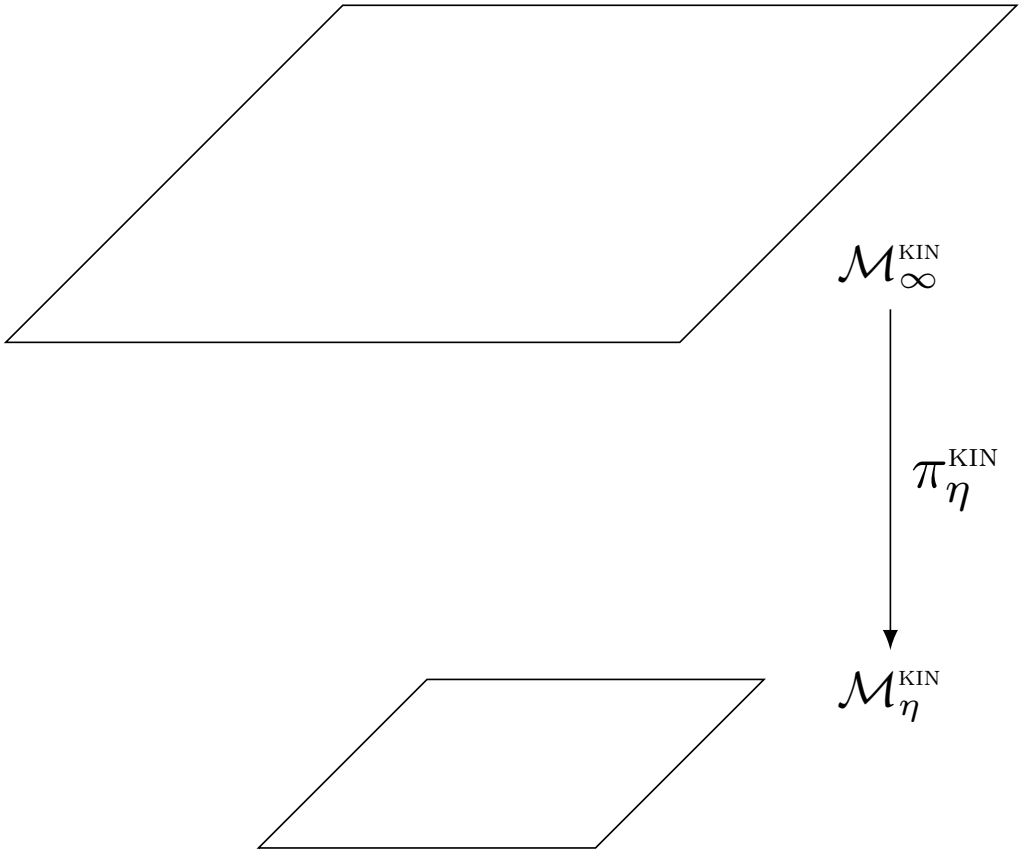
- each sector separately
- cross-sector correlations are lost
- not representable on the ITP:

$$\rho = \bigotimes_{f\in\Lambda} \left(\frac{|\Omega_f\rangle\langle\Omega_f| + |\Omega'_f\rangle\langle\Omega'_f|}{2} \right)$$

[ITP: von Neumann '39]

Implementation of Gauge Constraints and Dynamics

Nice Constraints



Restrictive requirements:

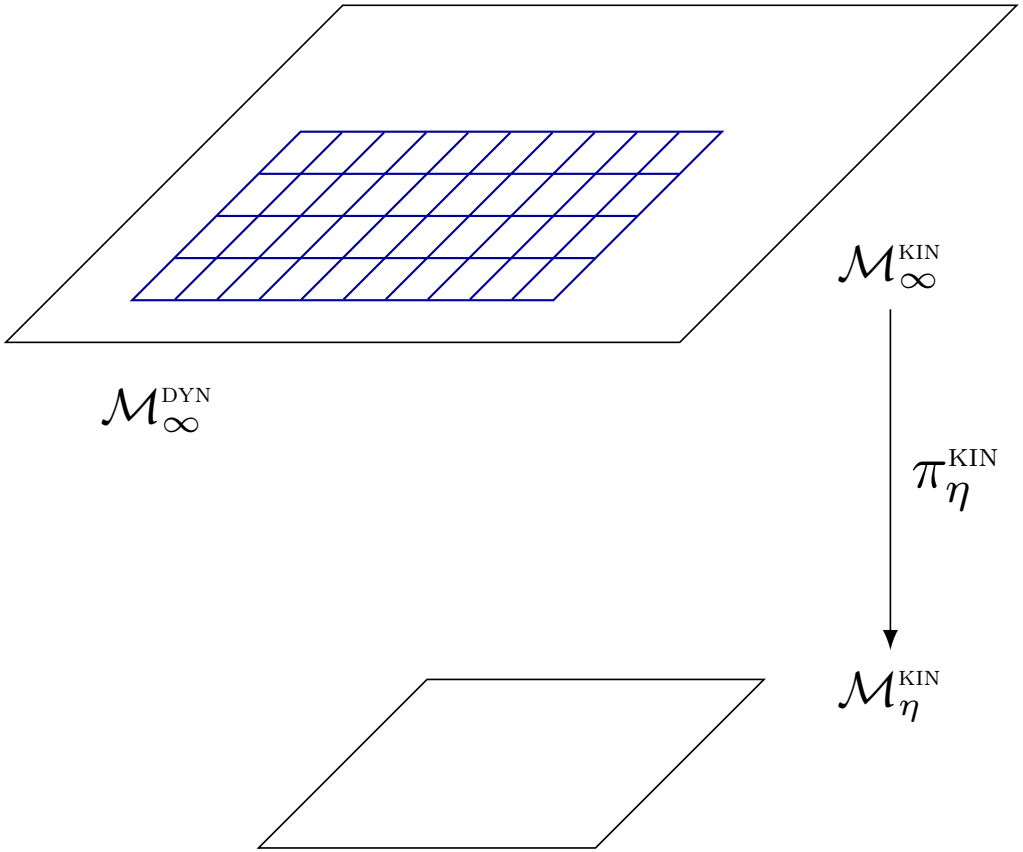
- orbits are projected on orbits $\rightarrow \pi_\eta^{\mathrm{DYN}}$ between reduced phase spaces
- compatible with symplect. structures

Dynamical projective system & transport maps:

- states to projective families of orbits
- observables

Implementation of Gauge Constraints and Dynamics

Nice Constraints



Restrictive requirements:

- orbits are projected on orbits $\rightarrow \pi_\eta^{\mathrm{DYN}}$ between reduced phase spaces
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Dynamical projective system & transport maps:

- states to projective families of orbits
- observables

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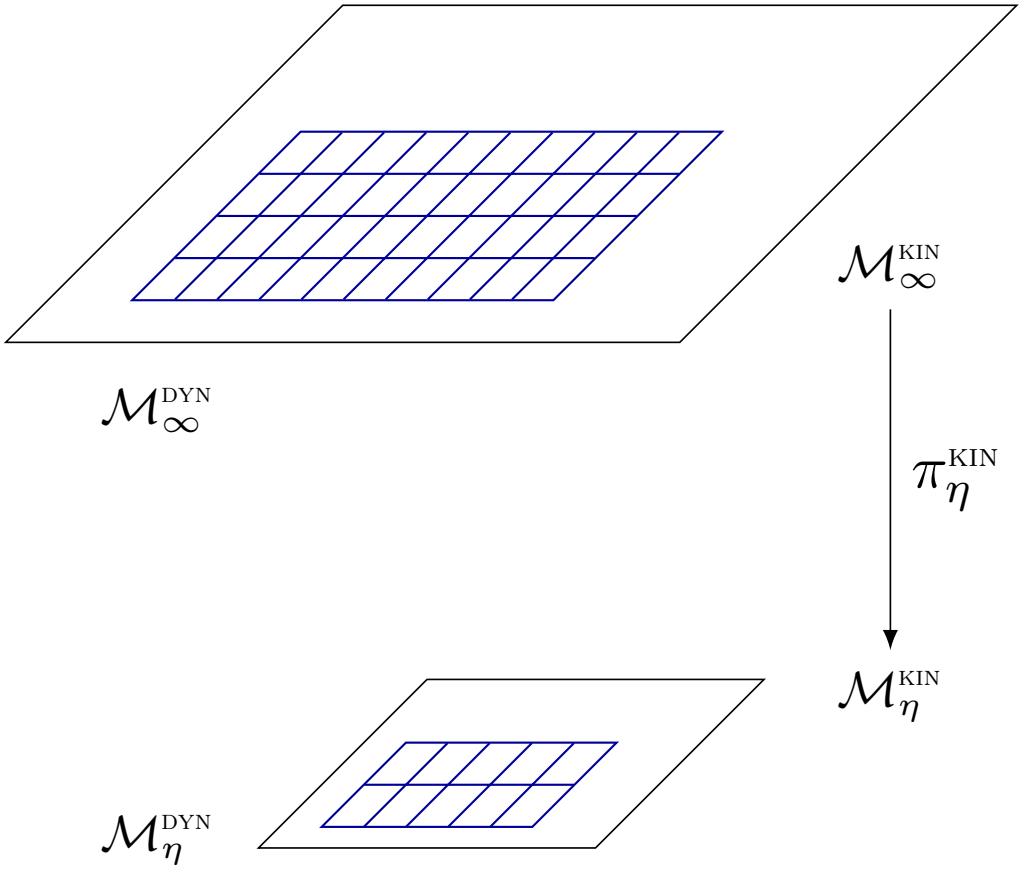

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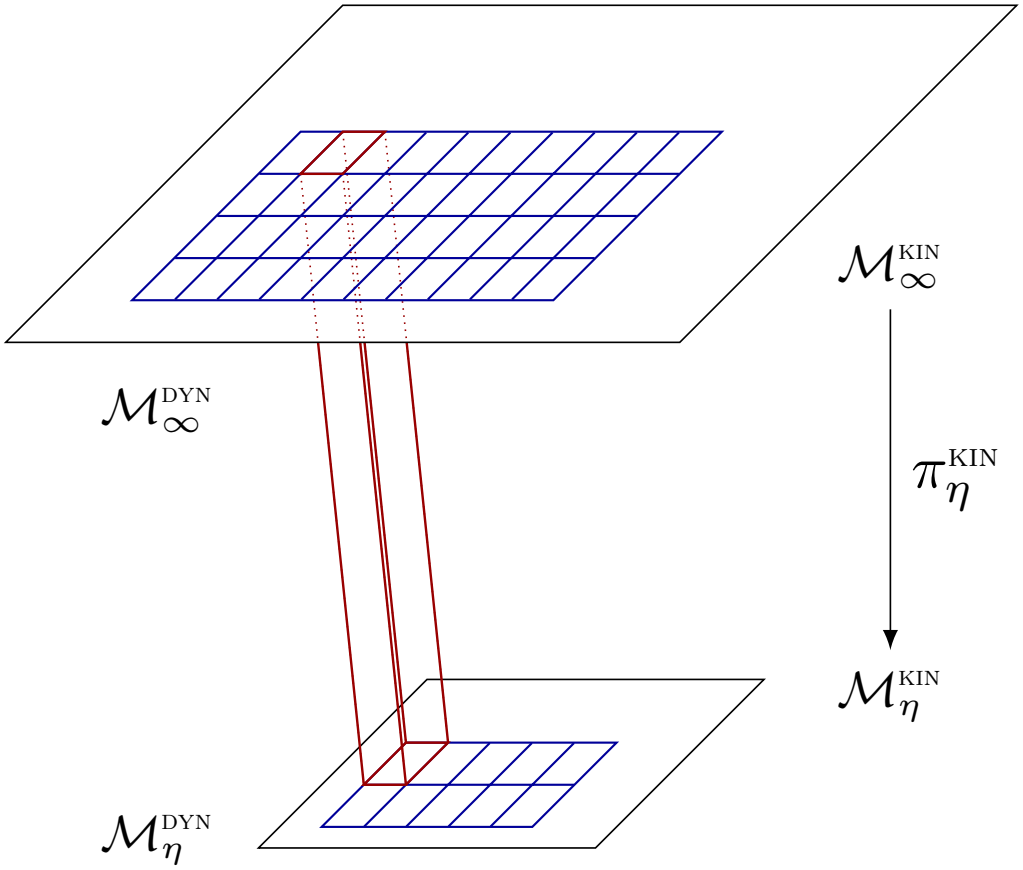

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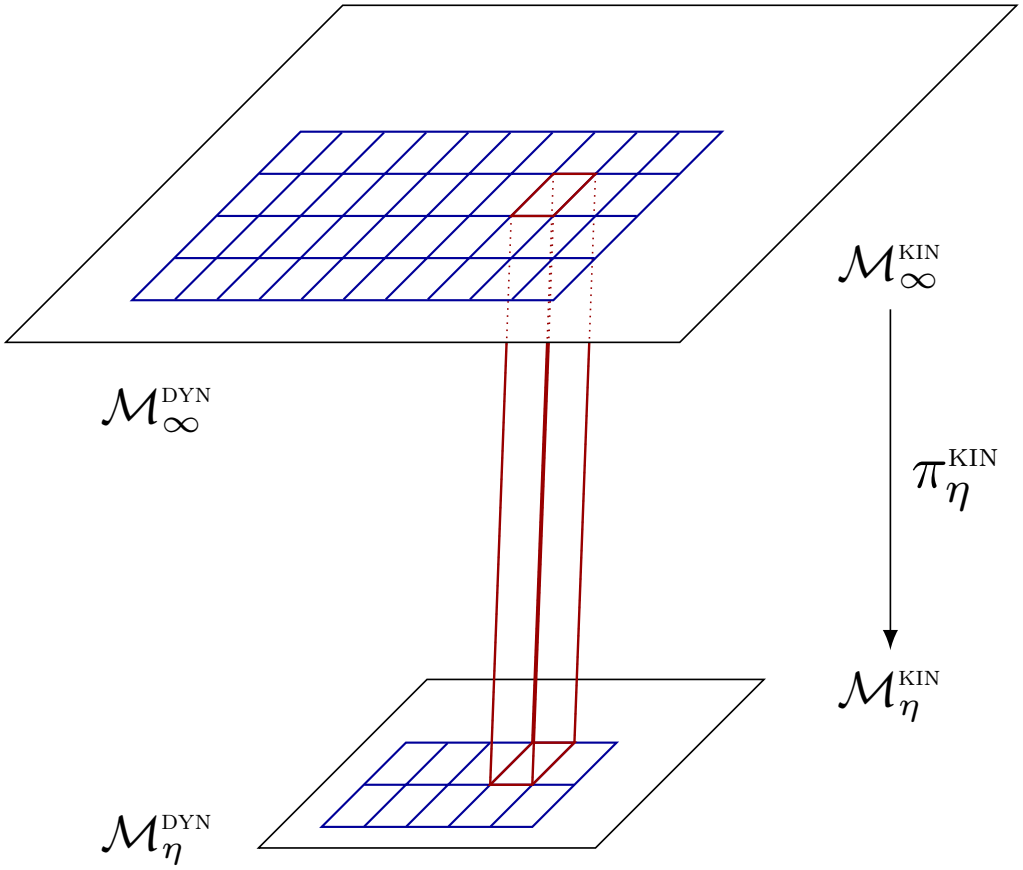

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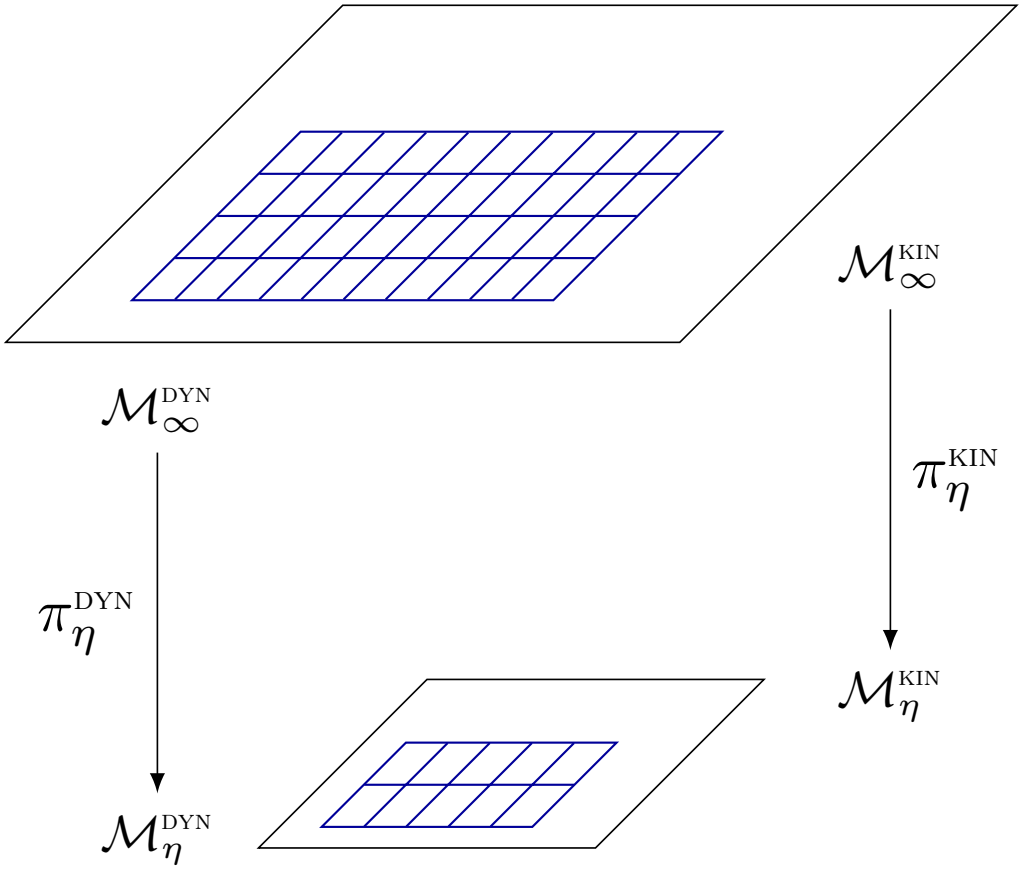

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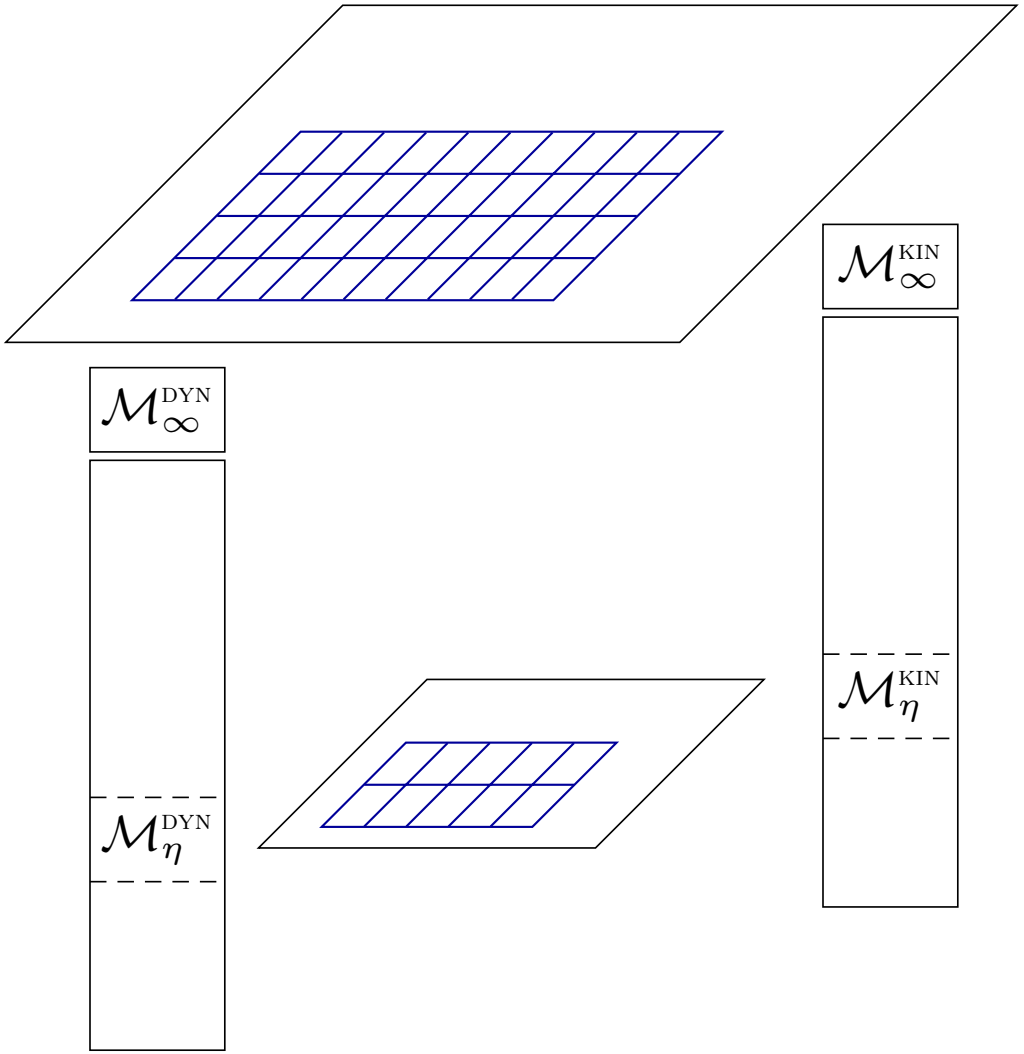

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Implementation of Gauge Constraints and Dynamics

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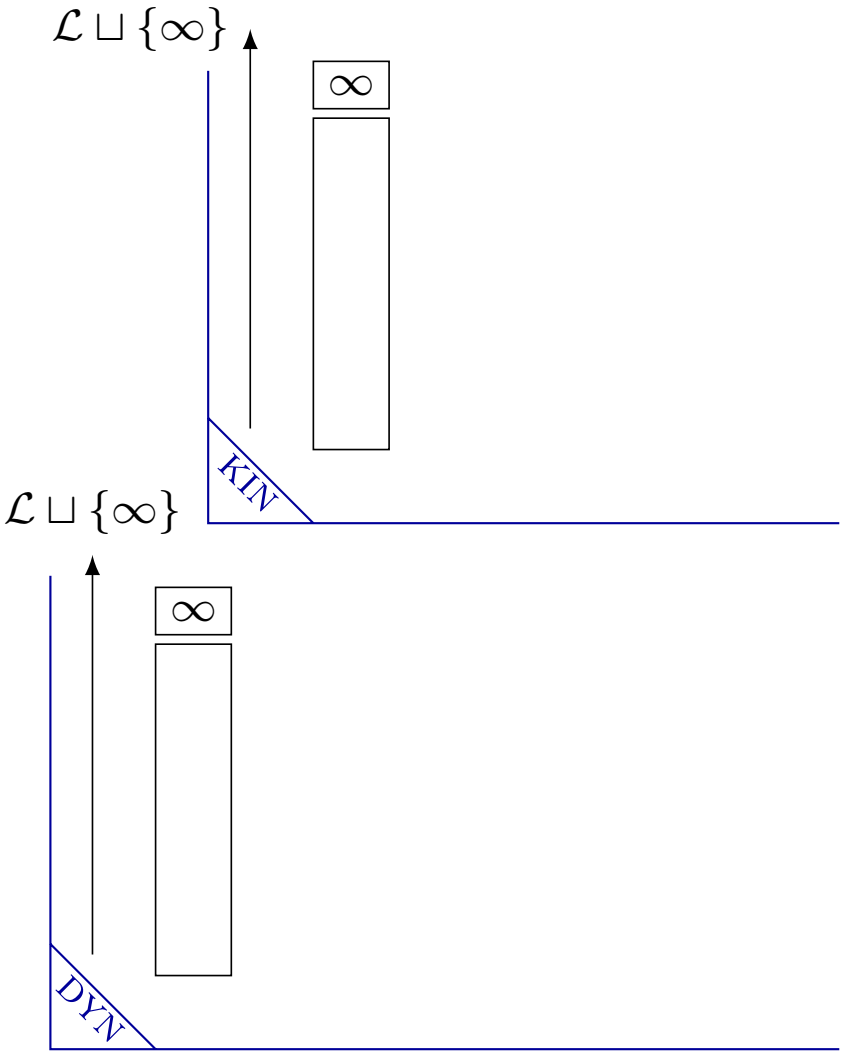

Successive approximations:

- labelled by $\varepsilon \in \mathcal{E}$
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Projections between approximated theories:

- dynamical projective system on a subset of $\mathcal{E} \times \mathcal{L}$
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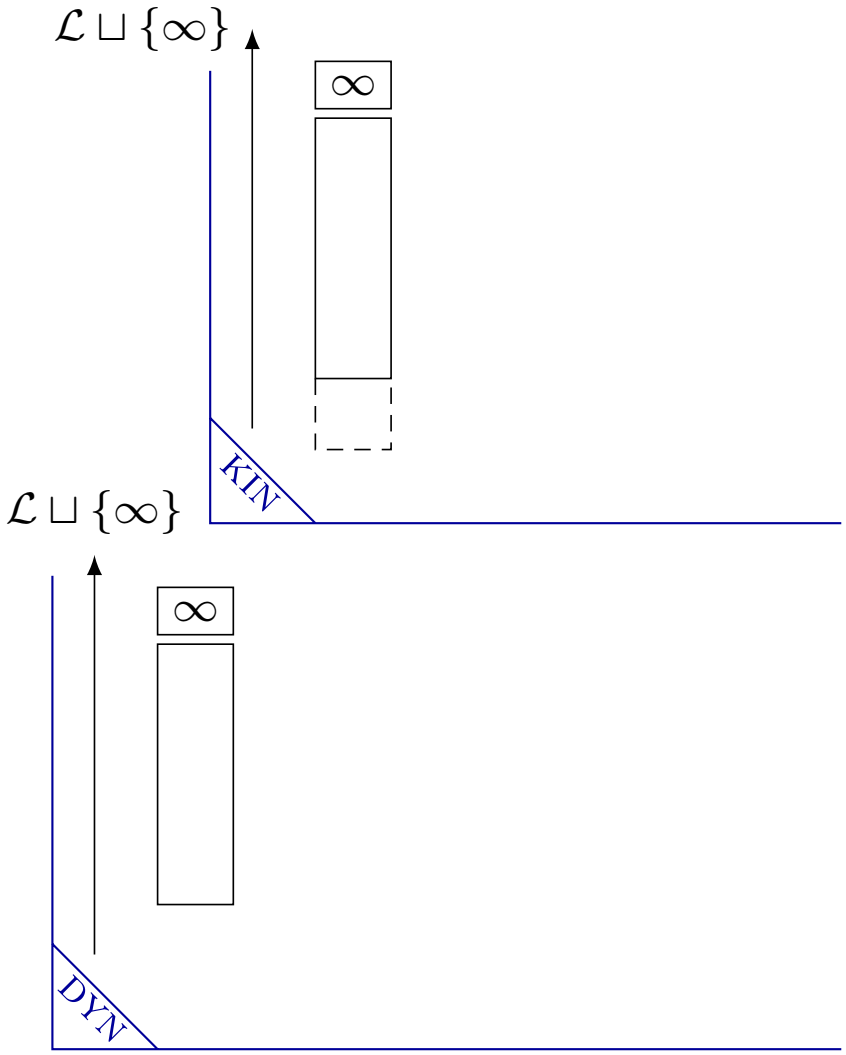
Implementation of Gauge Constraints and Dynamics

Unfitting Constraints



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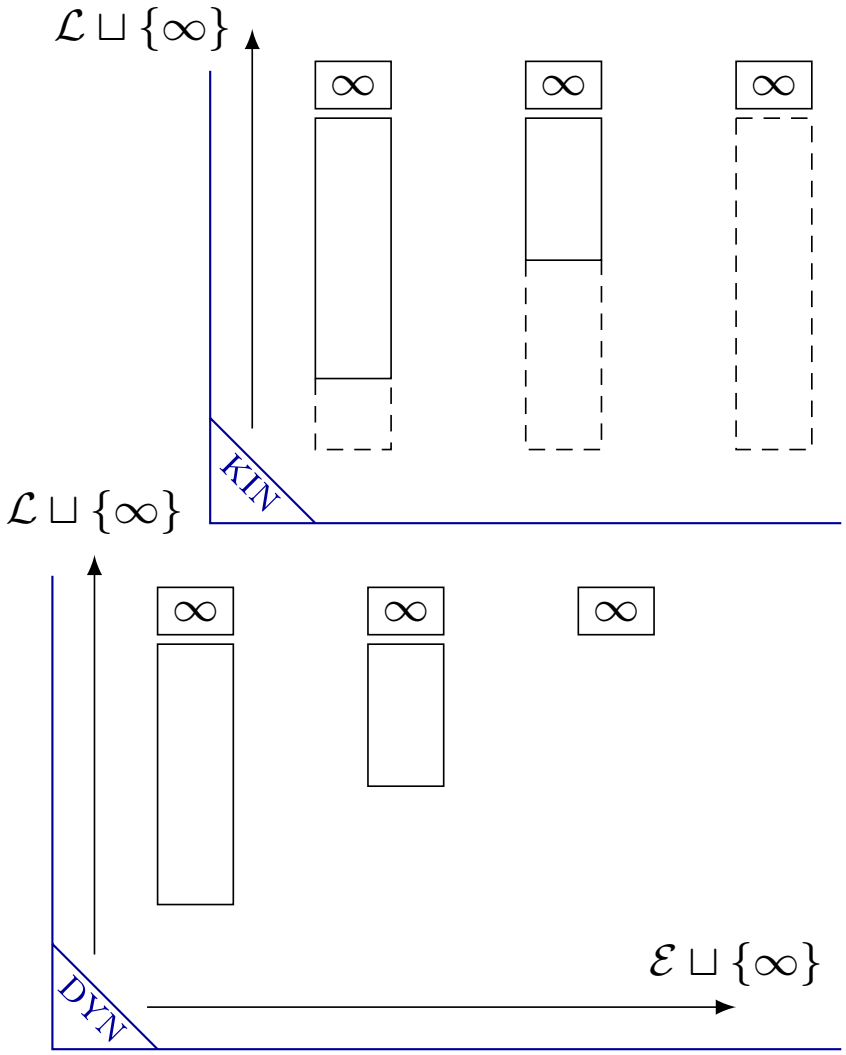

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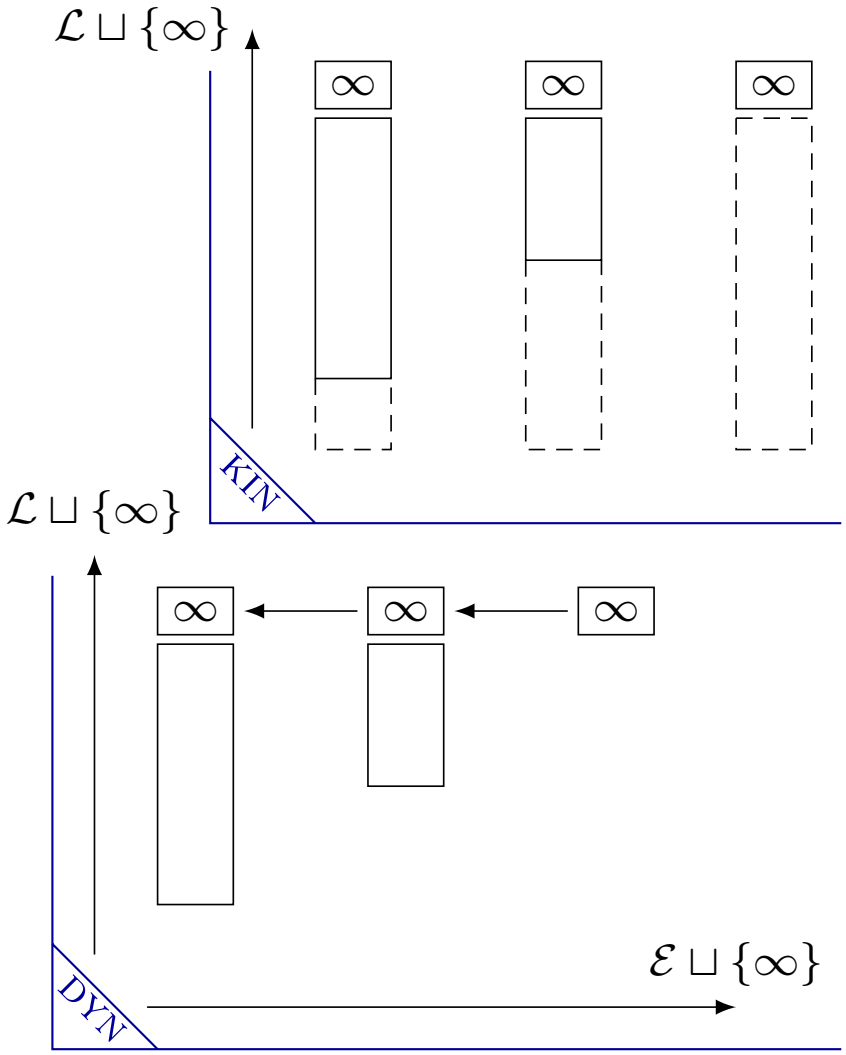

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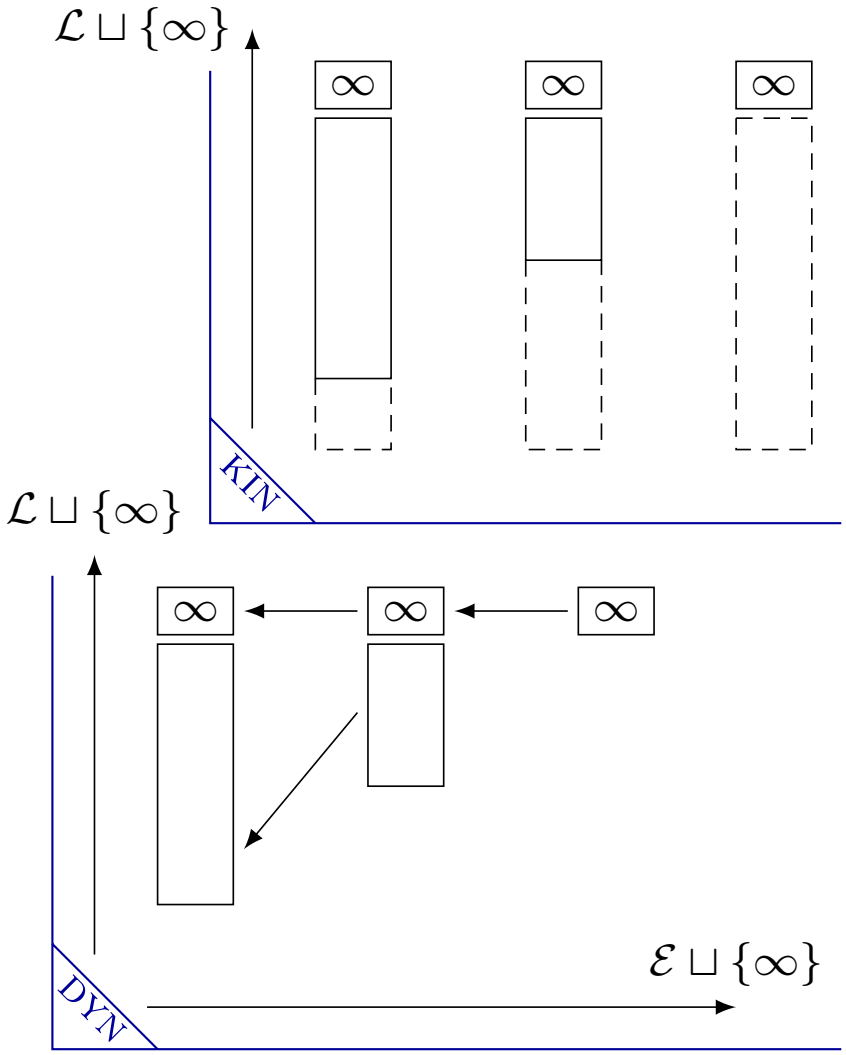

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Implementation of Gauge Constraints and Dynamics

Toy Model: Schrödinger Equation

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$$E - \langle \psi, H\psi \rangle = 0$$

$$\mathcal{M}_{\infty}^{\text{KIN}} = \mathcal{H}_{\text{one part.}} \times \mathbb{R}^2 = \{(\psi;\ t, E)\}$$

Approximations:

- $\epsilon > 0$ deformation $\rightarrow$ compact orbits
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Proof of principle for previous strategy:

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$\epsilon > 0, \quad I \subset \mathcal{H}_{\text{one part.}}$

$$\Big(E - \langle \psi,\, H_I\, \psi \rangle\Big)^2 + \epsilon^4\, t^2 = \epsilon^2$$

$$\& \qquad \psi \in I$$

where $H_I = \Pi_I\, H\, \Pi_I$

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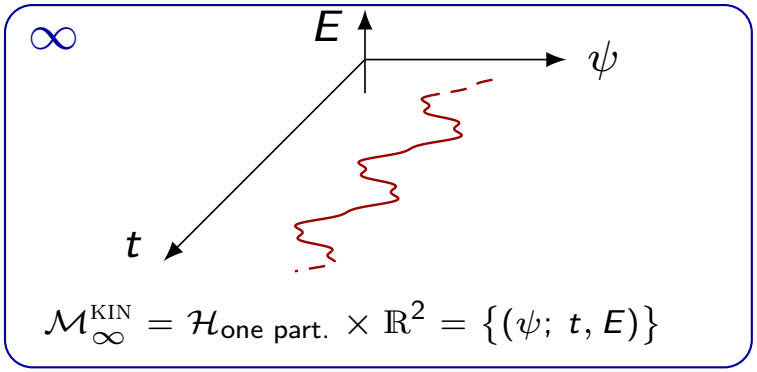

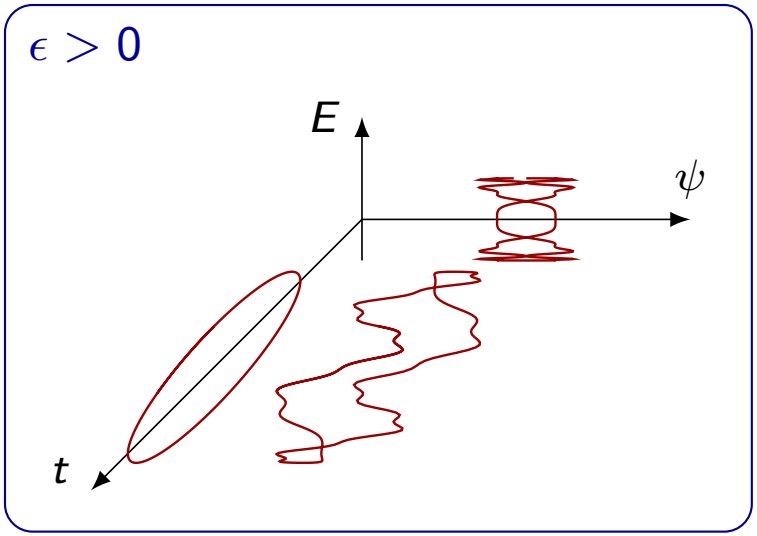

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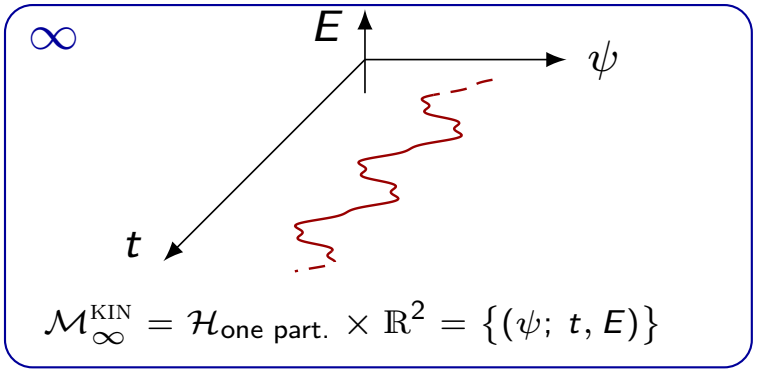

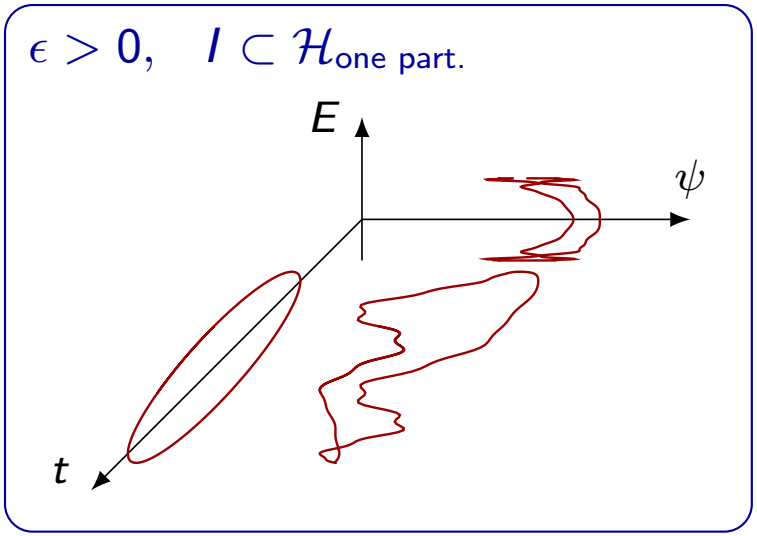

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Quantizing Gravity *

Perturbative Formulation ▸ $g = g_o + h$ ▸ perturb. QFT	
non- renormalizable	

* disclaimer: this table is **not** exhaustive!

[Non-renormalizability: Goroff & Sagnotti '85]

Quantizing Gravity *

Perturbative Formulation ▸ $g = g_o + h$ ▸ perturb. QFT		
non- renormalizable		
String Theory	Asymptotic Safety	

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[Strings: Green, Polyakov, Schwarz, Witten... '80s; Asympt. safe: Parisi, Weinberg, Wilson... '70s]

Quantizing Gravity *

Perturbative Formulation	Dynamical Triangulations	
▸ $g = g_o + h$ ▸ perturb. QFT	▸ discretization ▸ path-integral	
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[(C)DT: Ambjørn, Jurkiewicz, Loll... '90s]

Quantizing Gravity *

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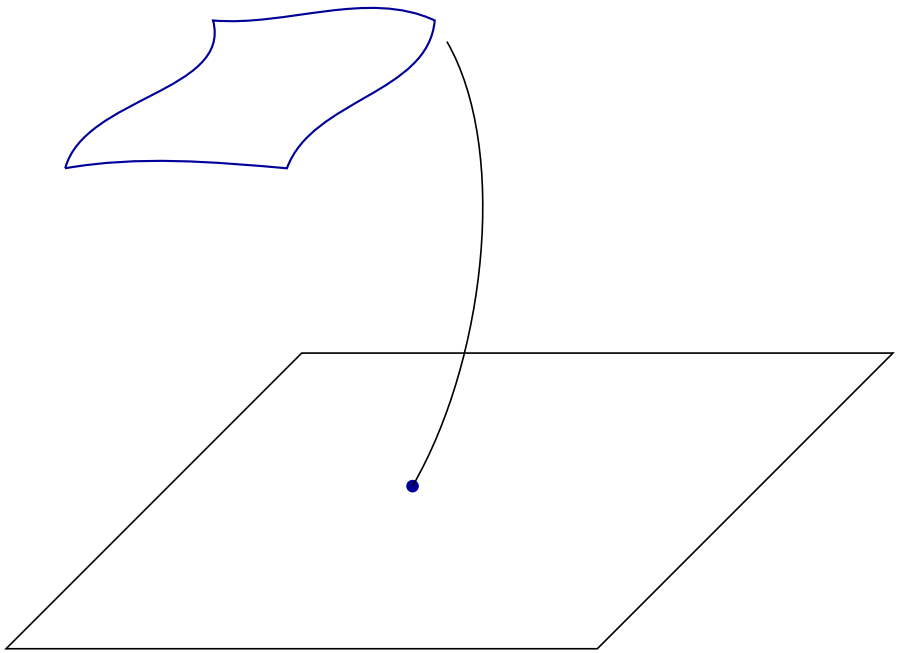
Canonical General Relativity

The ADM Formalism



[Canonical GR: Arnowitt, Deser, Misner '62]

Phase space of spacial slices:

- configuration variables: 3-geometry
- momentum variables: extrinsic curvature

Gauge constraints:

- 3-slice in constraint surface iff can be sliced out of Einstein 4-geometry
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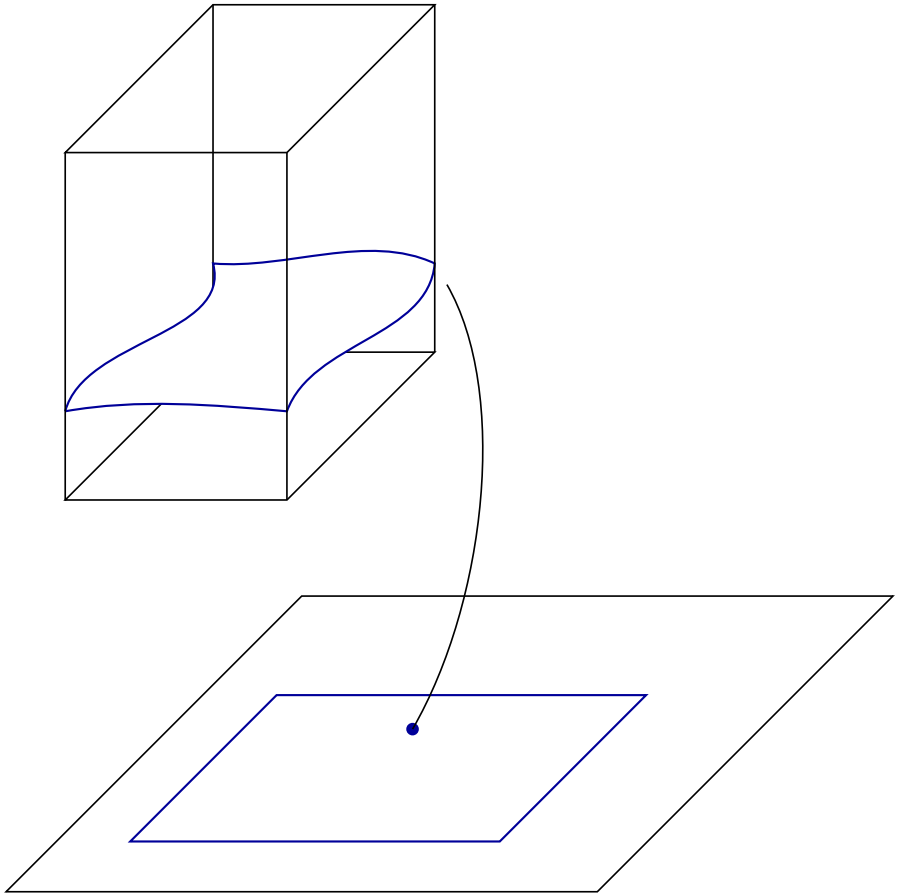

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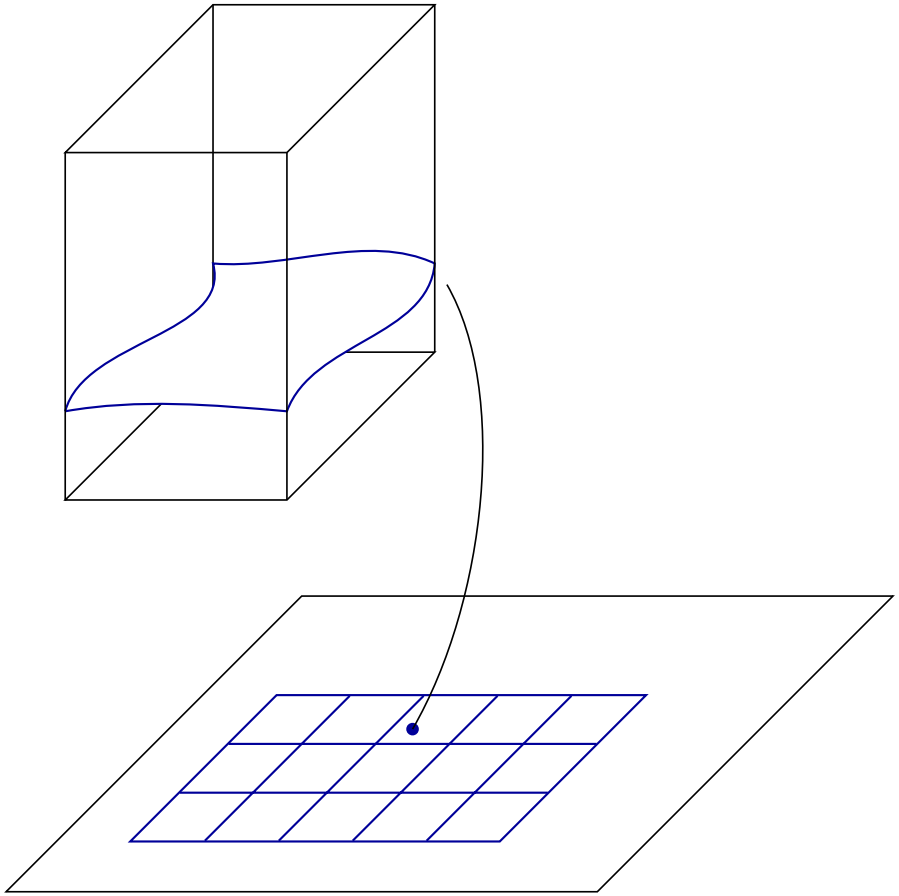

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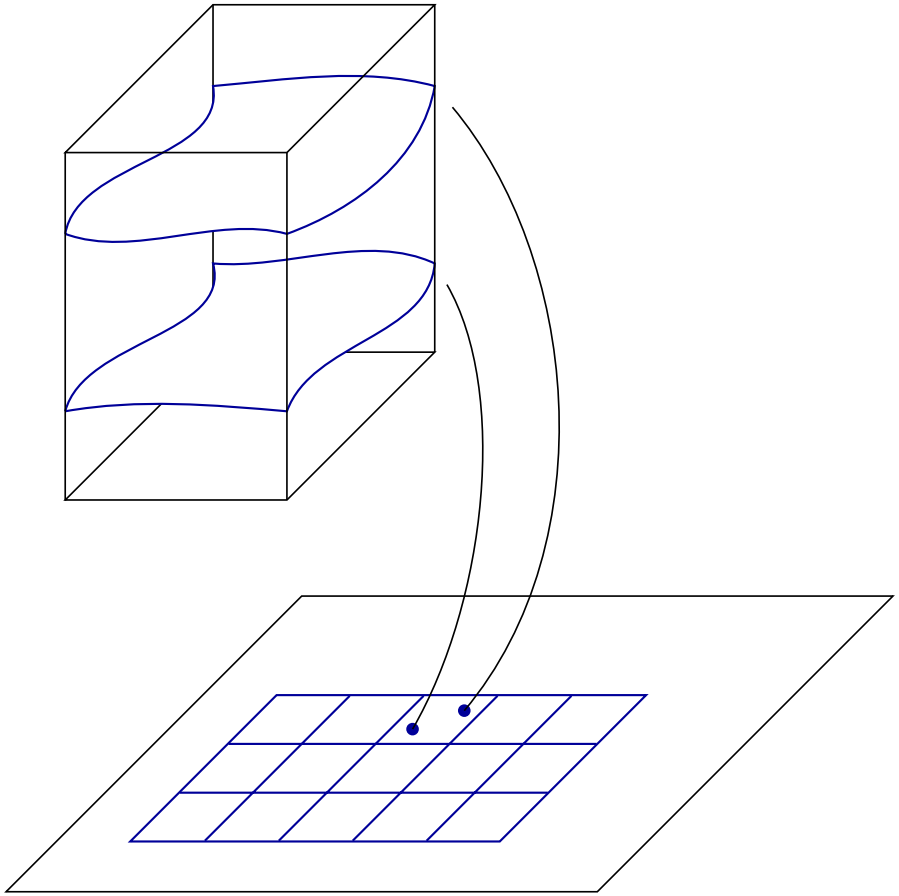

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String Theory	Asymptotic Safety	Causal Dyn. Triangl.	Spin Foams	Loop Quant. Gravity

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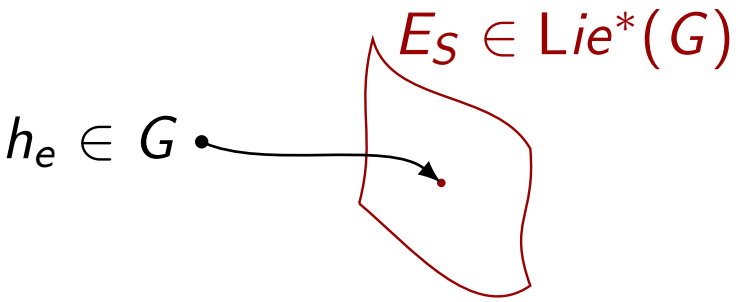
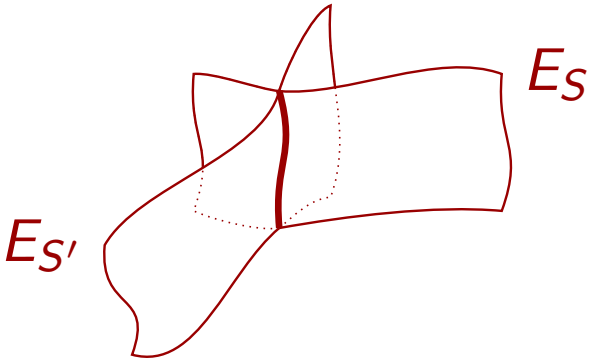
[LQG/Spin foams: Ashtekar, Lewandowski, Pullin, Rovelli, Smolin, Thiemann... '80s]

Loop Quantum Gravity

The Holonomy-Flux Variables



$$\{h_e\,,\,E_{S,i}\} = \tau_i\,.\,h_e$$



$$\{E_{S,i}\,,\,E_{S',j}\} = C^k_{ij}\;E_{S\cap S',k}$$

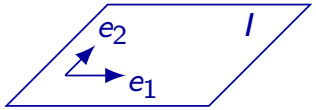
GR as a $G = \mathcal{SU}(2)$ gauge theory:

- $\mathrm{Lie}^*(G)$-valued **fluxes** encode spacial 3-geometry
- G-valued **holonomies** combine intrinsic and extrinsic curvature
- Poisson algebra can be regularized, no background metric needed
- reduces to ADM formalism once additional G-gauge invariance is imposed

[Holonomy-flux algebra: Ashtekar, Isham, Rovelli, Smolin, Lewandowski, Pullin, Gambini,...]

Loop Quantum Gravity

The Ashtekar–Lewandowski Hilbert Space

Fock Space	
 $I = \text{span}\{e_1,\ e_2\}$ $\subset \mathcal{H}_{\text{one particule}}$	
$\mathcal{F}_I = L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ $= \text{span}\{\ \lvert n_1,\ n_2\rangle \mid n_1,\ n_2 \in \mathbb{N}\}$	
$\widehat{\mathrm{a}}_{e_1}\ \lvert n_1,\ n_2\rangle = \sqrt{n_1}\ \lvert n_1 - 1,\ n_2\rangle$ $\widehat{\mathrm{a}}^\dagger_{e_1}\ \lvert n_1,\ n_2\rangle = \sqrt{n_1 + 1}\ \lvert n_1 + 1,\ n_2\rangle$	

[LQG Hilbert space: Isham, Ashtekar, Lewandowski,... '92–95]

Loop Quantum Gravity

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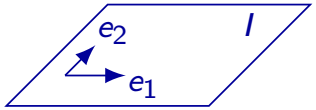
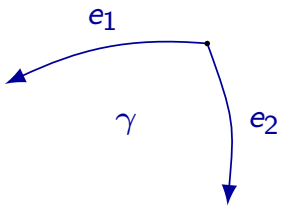
 $I = \text{span}\{e_1,\, e_2\}$ $\subset \mathcal{H}_{\text{one particule}}$	 $\gamma = (e_1,\, e_2)$ on the spacial slice Σ
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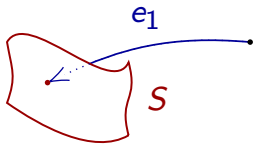
$\widehat{\mathrm{a}}_{e_1} \lvert n_1,\, n_2\rangle = \sqrt{n_1}\, \lvert n_1 - 1,\, n_2\rangle$ $\widehat{\mathrm{a}}^\dagger_{e_1} \lvert n_1,\, n_2\rangle = \sqrt{n_1 + 1}\, \lvert n_1 + 1,\, n_2\rangle$	$\widehat{f \circ h}_{e_1} \lvert\psi_\gamma\rangle = \lvert f(g_1)\, \psi_\gamma\rangle$ $\widehat{E}_{S,i} \lvert\psi_\gamma\rangle = \lvert {}_i\nabla^{[g_1]}_{\tau_i}\, \psi_\gamma\rangle$ 

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Loop Quantum Gravity

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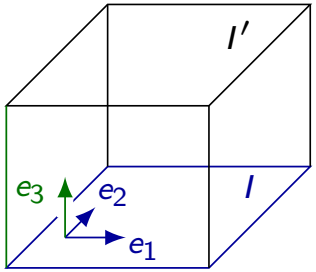
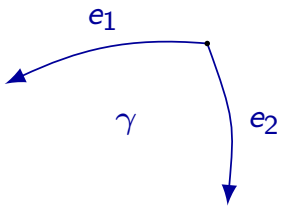
 $I' = \text{span}\{e_1,\ e_2,\ e_3\}$ $I = \text{span}\{e_1,\ e_2\}$ $\subset \mathcal{H}_{\text{one particule}}$	 $\gamma = (e_1,\ e_2)$ on the spacial slice Σ
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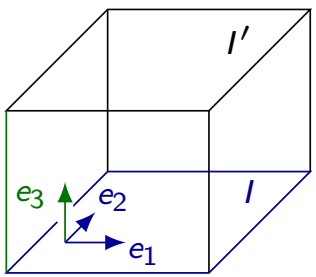
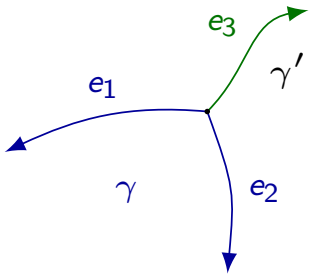
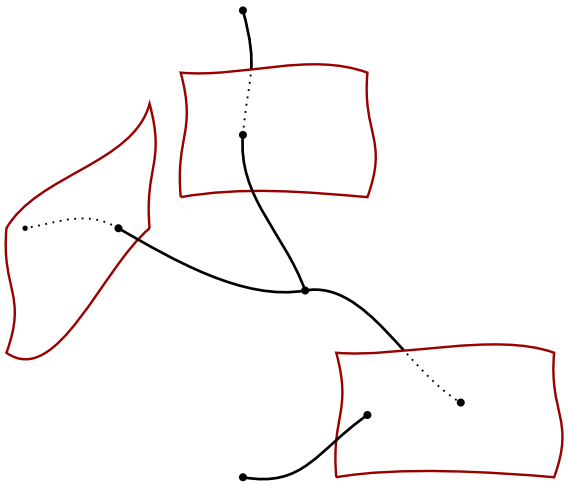
The Ashtekar–Lewandowski Hilbert Space

Fock Space	AL Hilbert Space
 $I' = \mathrm{span}\{e_1, e_2, e_3\}$ $I = \mathrm{span}\{e_1, e_2\}$ $\subset \mathcal{H}_{\text{one particule}}$	 $\gamma' = (e_1, e_2, e_3)$ $\gamma = (e_1, e_2)$ on the spacial slice Σ
$\mathcal{F}_I = \mathrm{span}\{\, \lvert n_1, n_2\rangle \mid n_1, n_2 \in \mathbb{N}\}$	$\mathcal{H}_\gamma = \{\, \lvert\psi_\gamma\rangle \mid \psi_\gamma : g_1, g_2 \mapsto \psi_\gamma(g_1, g_2)\}$
$\mathcal{F}_I \subset \mathcal{F}_{I'}$ $\lvert n_1, n_2\rangle \simeq \lvert n_1, n_2, 0\rangle$ $\mathcal{F}_{\text{Fock}} = \lim\limits_{I \subset \mathcal{H}_{\text{one part.}}} \mathcal{F}_I$	$\mathcal{H}_\gamma \subset \mathcal{H}_{\gamma'}$ $\lvert\psi_\gamma\rangle \simeq \lvert\psi_{\gamma'} : g_1, g_2, g_3 \mapsto \psi_\gamma(g_1, g_2)\rangle$ $\mathcal{H}_{\text{AL}} = \lim\limits_{\gamma \text{ on } \Sigma} \mathcal{H}_\gamma$

[LQG Hilbert space: Isham, Ashtekar, Lewandowski,... '92–95]

Projective State Space for LQG

The Label Set



The label set:

- a graph = a choice of holonomies
- a set of surfaces dual to this graph = a choice of conjugate fluxes
- the label set must be directed $\Rightarrow$ allow intersecting surfaces
- each label = a $\{\cdot,\cdot\}$-subalgebra $\Rightarrow$ need edges probing any intersection

[See also: Okołów '13 (Abelian case)]

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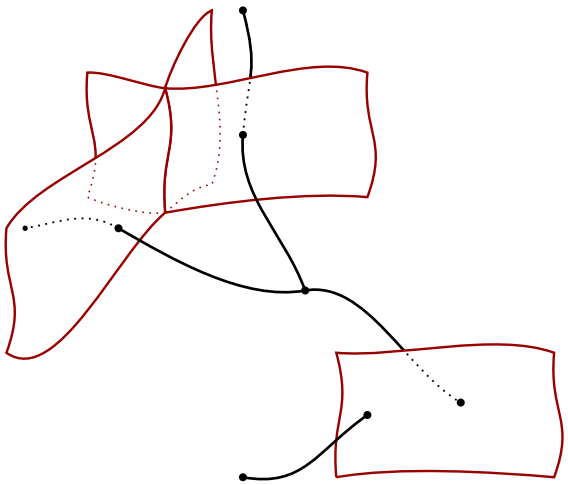

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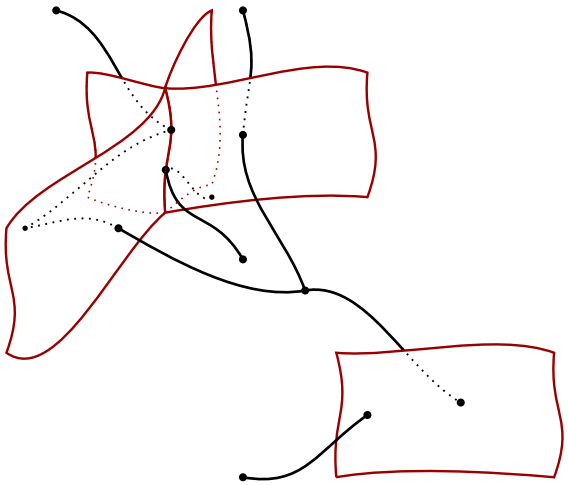

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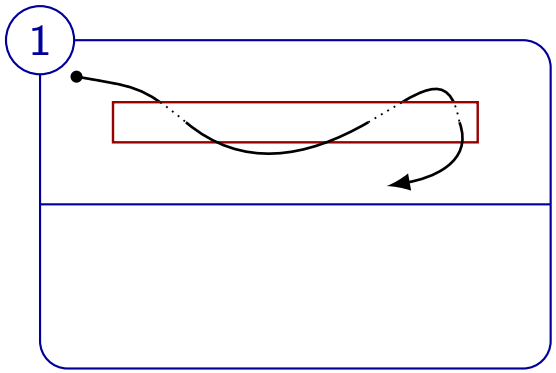
[See also: Okołów '13 (Abelian case)]

Holonomy-Flux Algebra

Directedness of the Label Set ($\forall \eta, \eta' \in \mathcal{L}$, $\exists\, \eta'' \succcurlyeq \eta, \eta'$)



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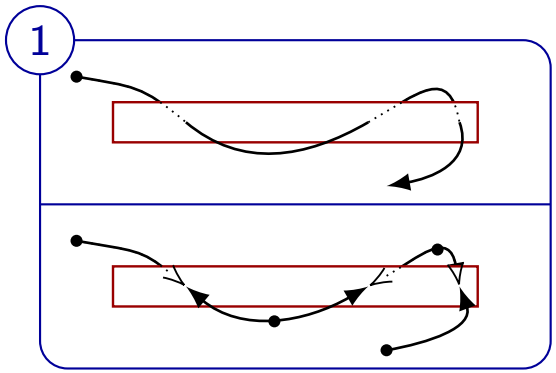

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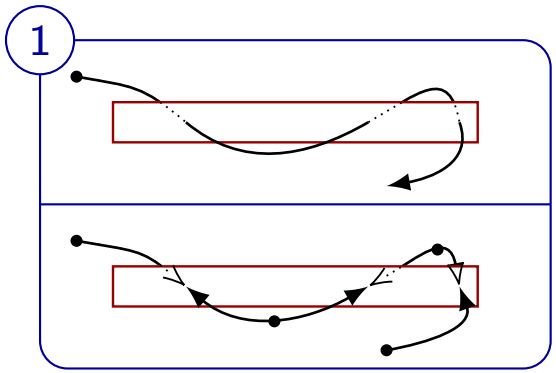

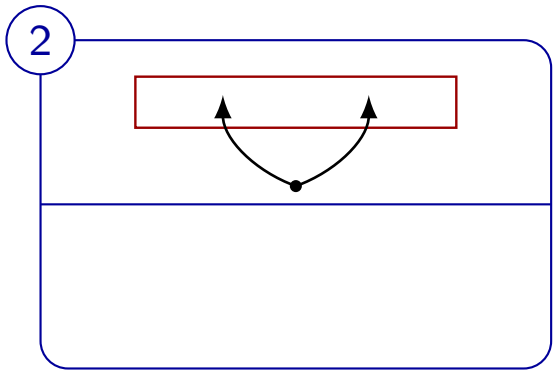

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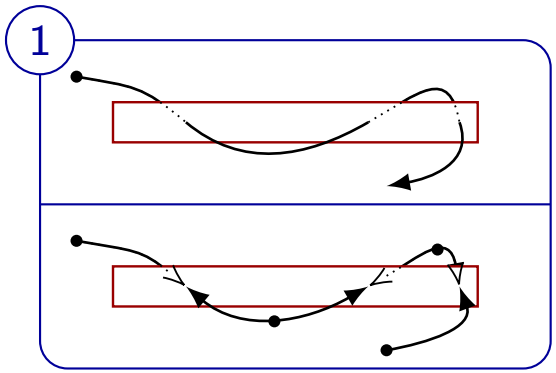

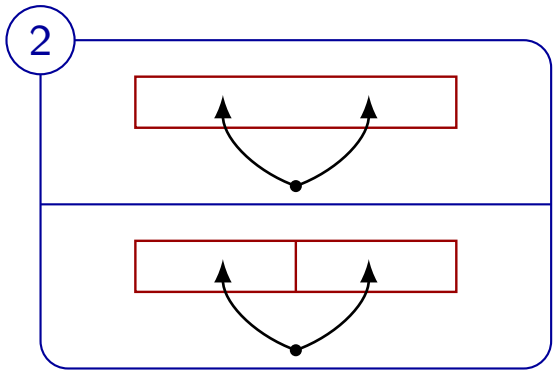

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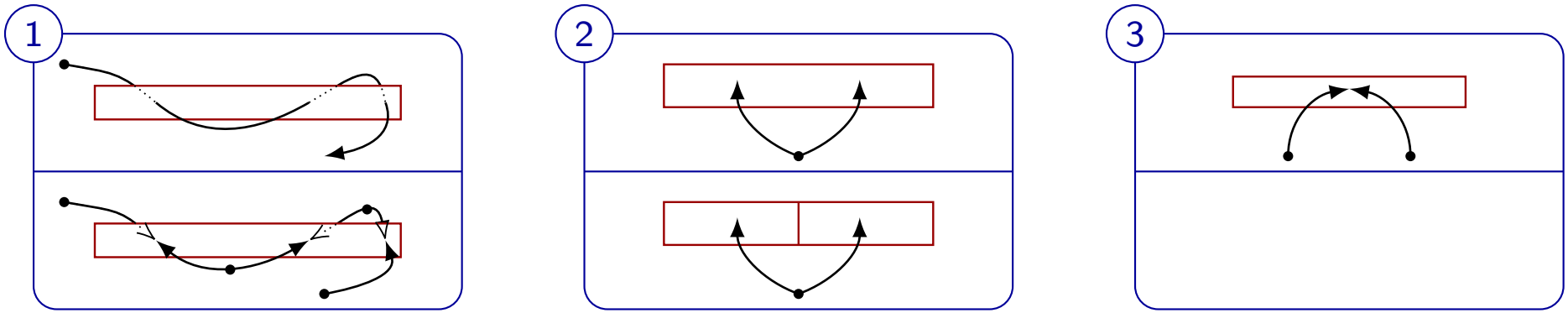

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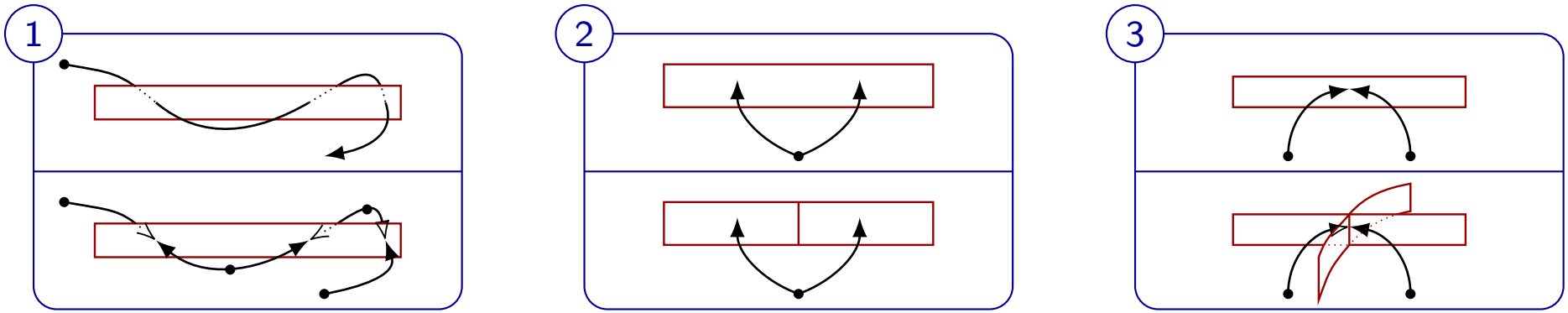

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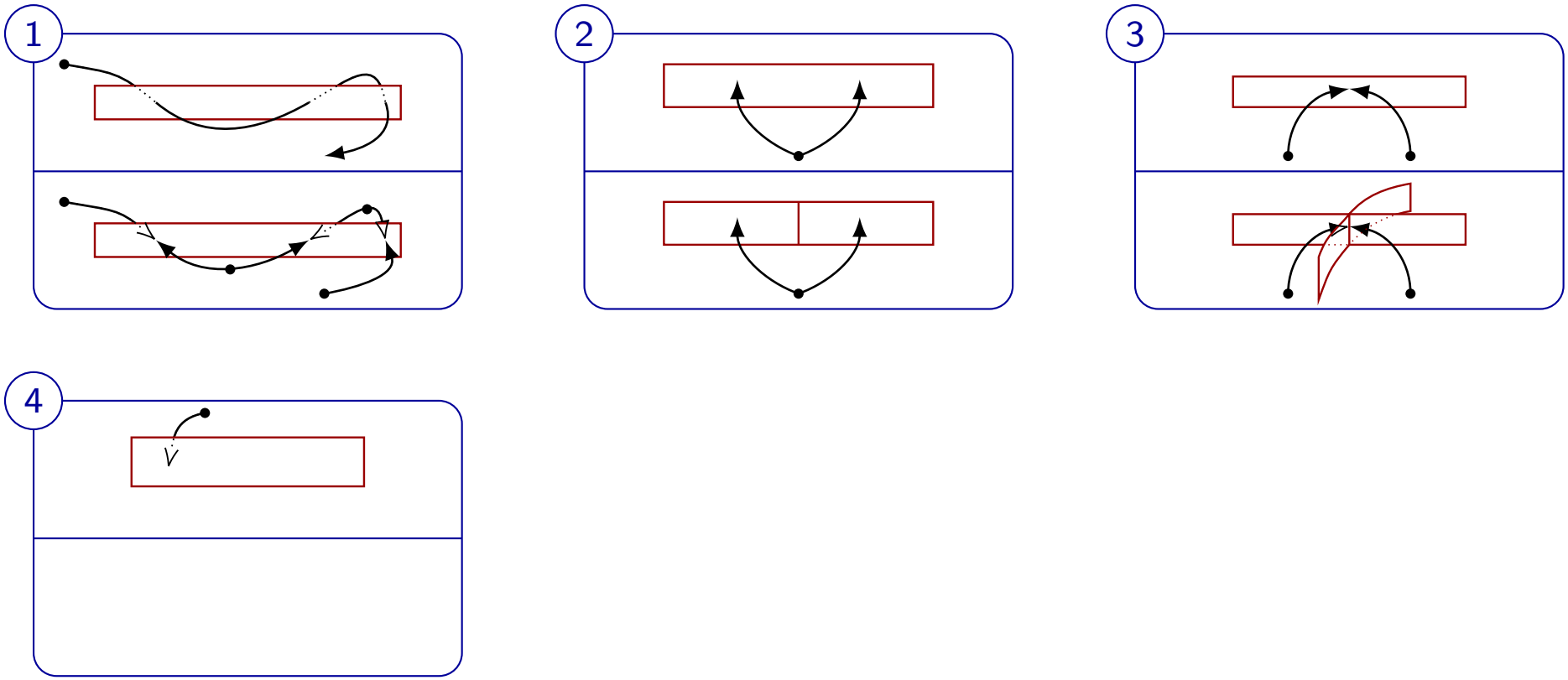

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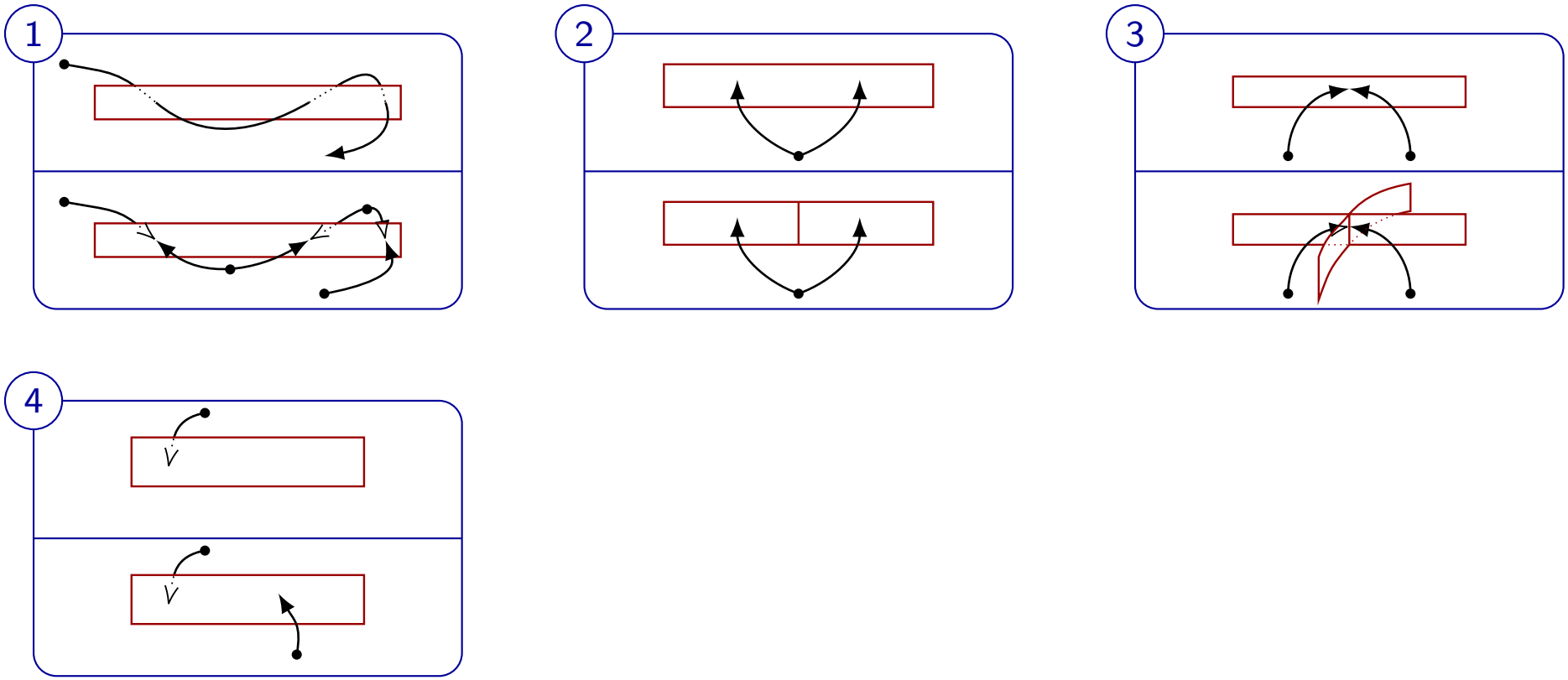

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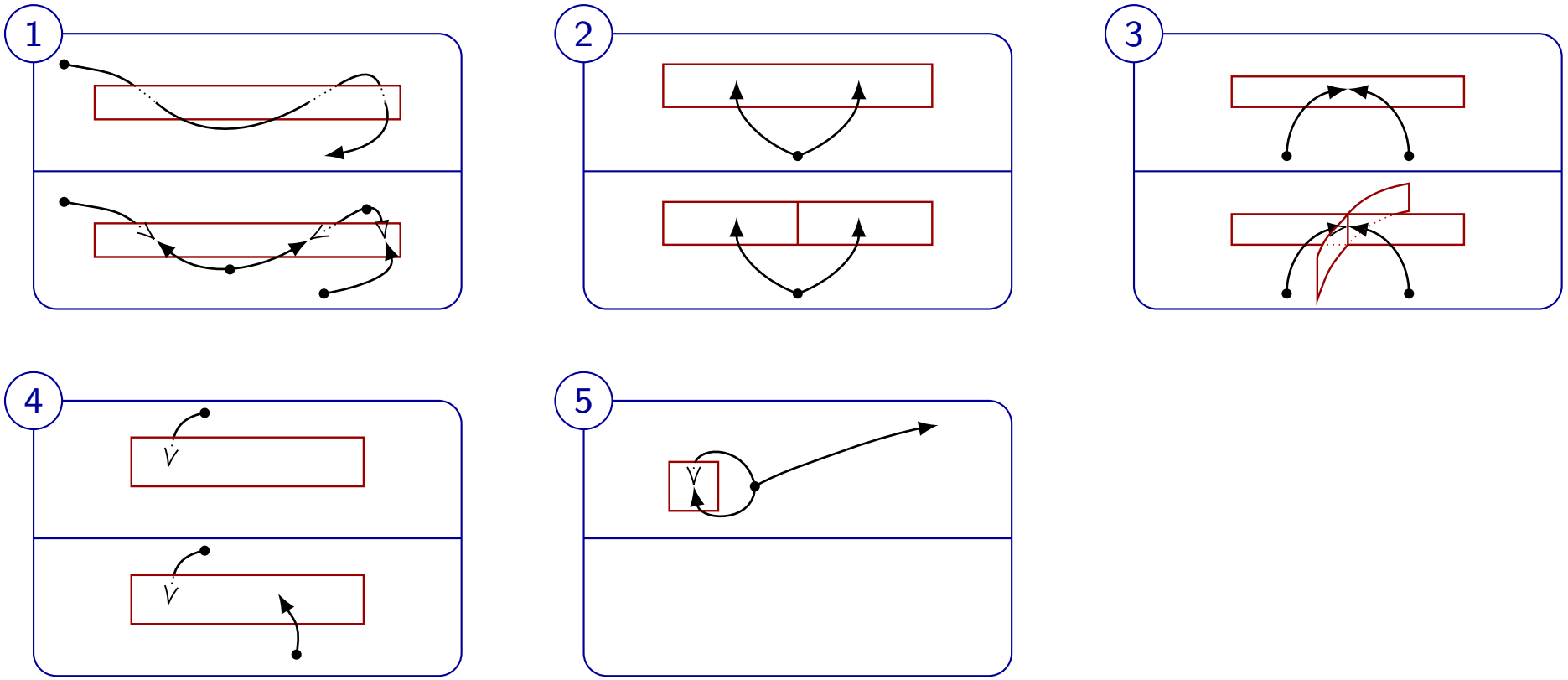

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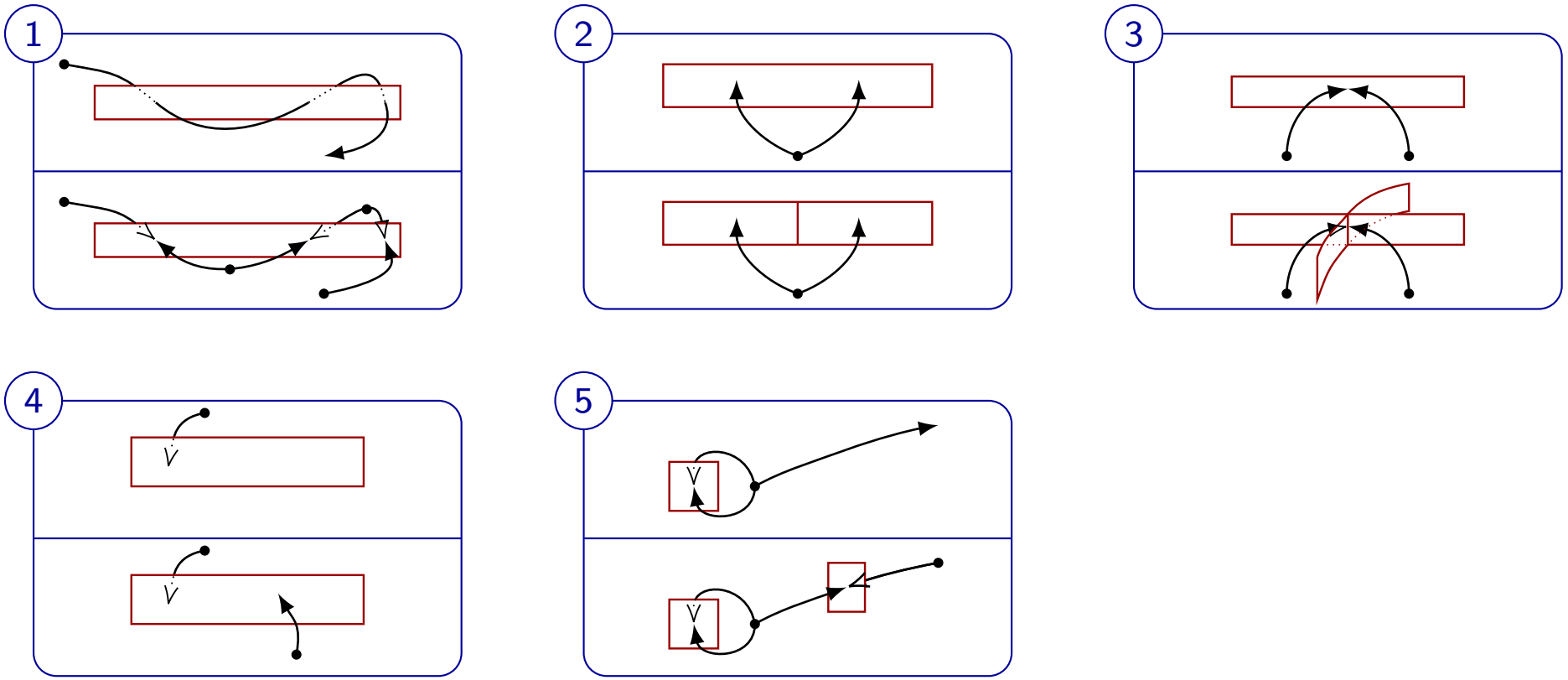
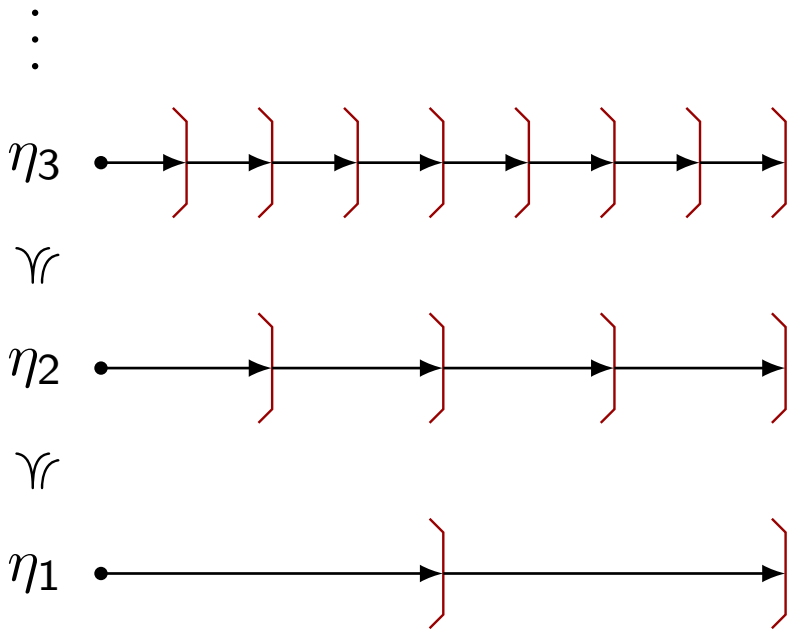


Universal Sequences: A 1-dimensional Model

Selection of an Increasing Sequence



Simplified version of holonomy-flux algebra:

- one-dimensional
- one-sided fluxes, acting only on the left

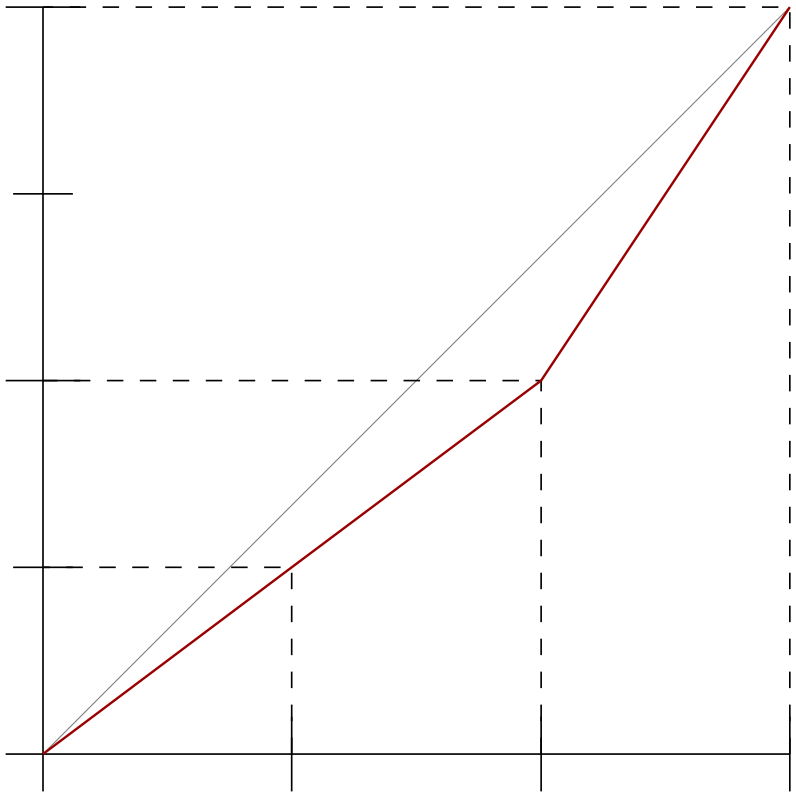
Selecting an increasing $(\eta_k)_{k\in\mathbb{N}}$:

- $\eta_k = \left\{ e_{\ell/2^k},\, S_{\ell/2^k} \,\middle|\, 1 \leqslant \ell \leqslant 2^k \right\}$
- $k \leqslant k' \Rightarrow \eta_k \preccurlyeq \eta_{k'}$

[See also: combinatorial LQG: Zapata '97; discrete quantum gravity: Gambini & Pullin;...]

Universal Sequences: A 1-dimensional Model

Universality and Approximation of Diffeomorphisms



One can formulate generic properties that $\left(\eta_k\right)_k$ should satisfy.

All sequences with these properties are equivalent:

- deformation mapping eg. $\left\{n/3^m\right\}$ to $\left\{\ell/2^k\right\}$, arbitrarily close to identity
- $\Rightarrow$ identification of the corresponding state spaces
- similarly: approximating diffeomorphisms

[$\neq$ universality at diffeo.-inv. level: Baez & Sawin '95, Zapata '97,...]

Universal Sequences: A 1-dimensional Model

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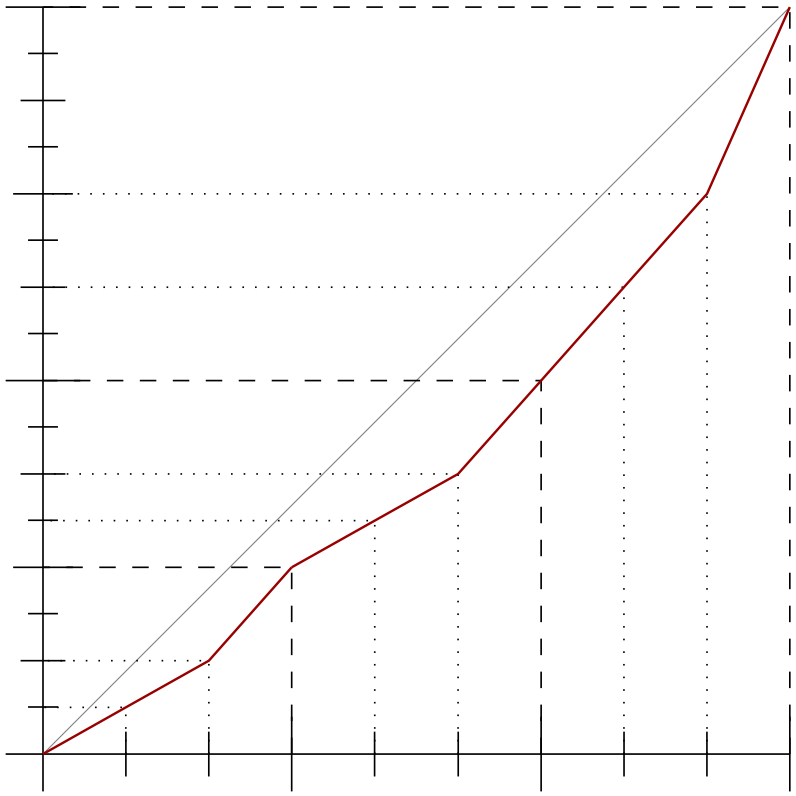

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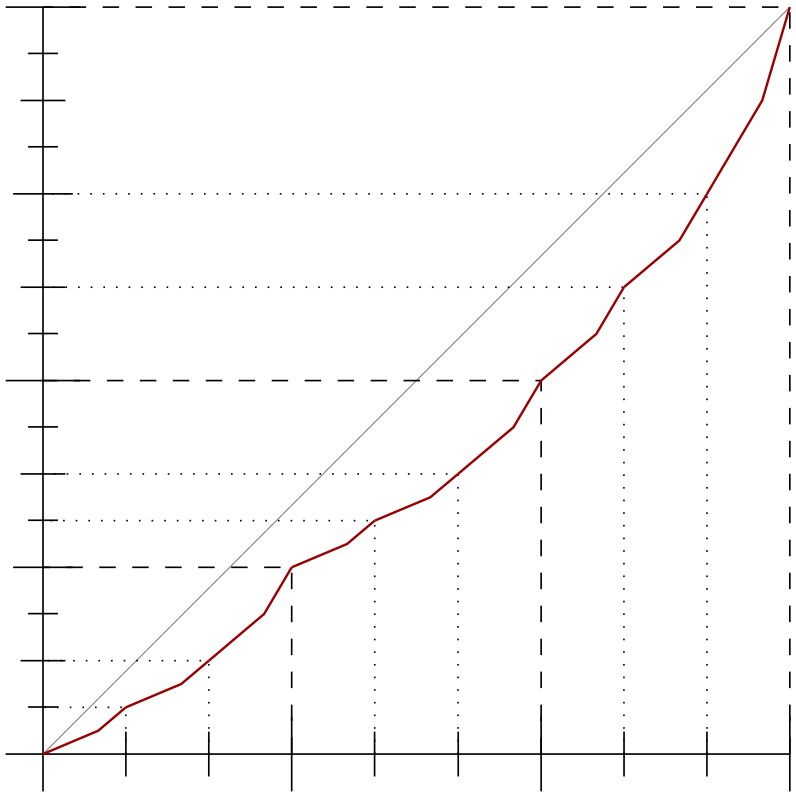

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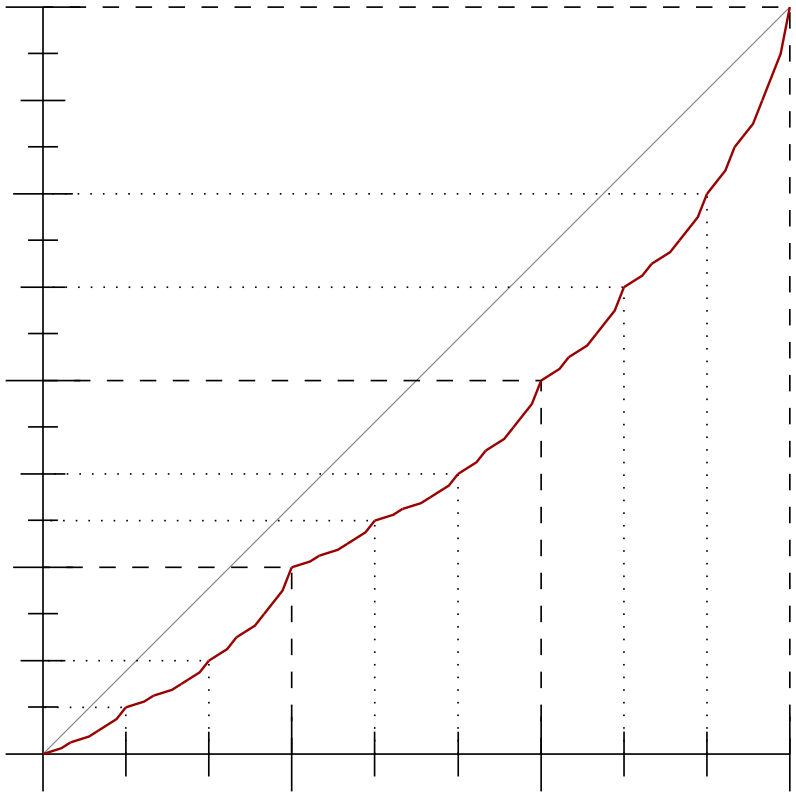

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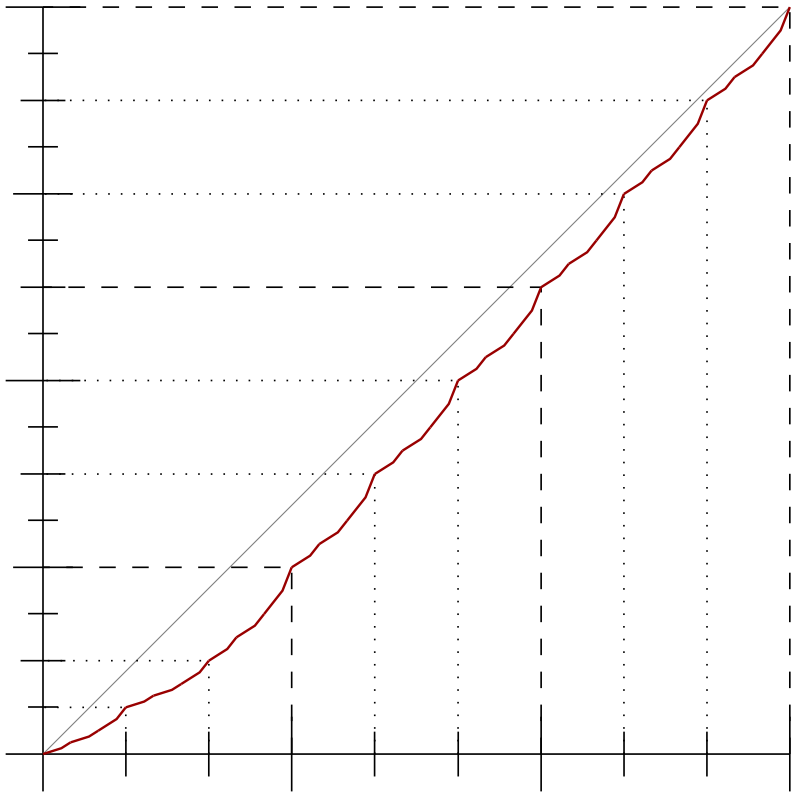

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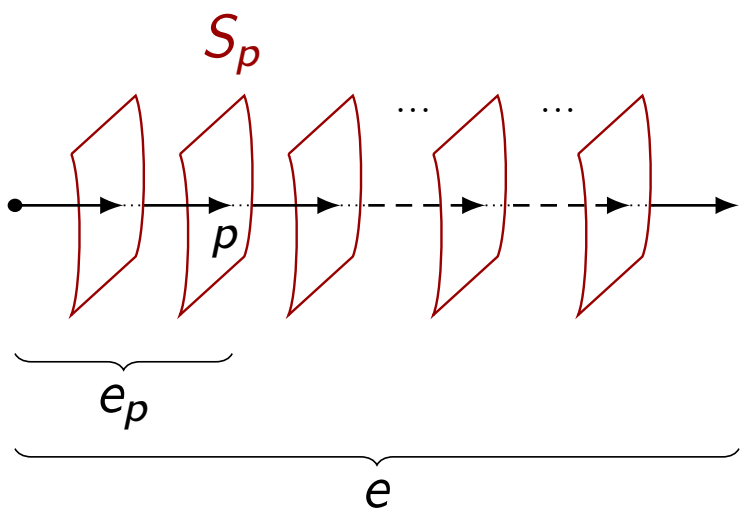
Obstruction to Finite Variances in the $G = \mathbb{R}$ Case

Selecting a Set of Observables with Uniformly Bounded Variances



$G = \mathbb{R}$

Looking for a state such that:

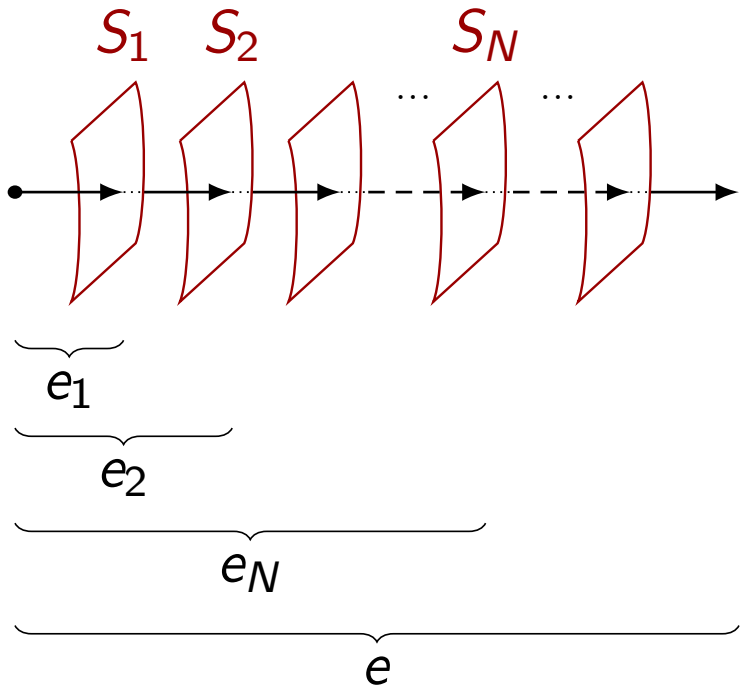
- $\forall e,\ \Delta \mathrm{h}_e^2 < \infty$
- $\forall S,\ \Delta \mathrm{E}_S^2 < \infty$

$\{p \in e\}$ uncountable:

- $\exists A /$
  $\{p \mid \Delta \mathrm{h}_{e_p}^2 \leqslant A\}$ uncountable
- $\exists A,\ B /$
  $\{p \mid \Delta \mathrm{h}_{e_p}^2 \leqslant A,\ \Delta \mathrm{E}_{S_p}^2 \leqslant B\}$
  uncountable

Obstruction to Finite Variances in the $G = \mathbb{R}$ Case

Bound from Heisenberg Uncertainty Relations



$$\Delta \mathrm{h}_{e_k}^2 \leqslant A, \quad \Delta \mathrm{E}_{S_k}^2 \leqslant B$$

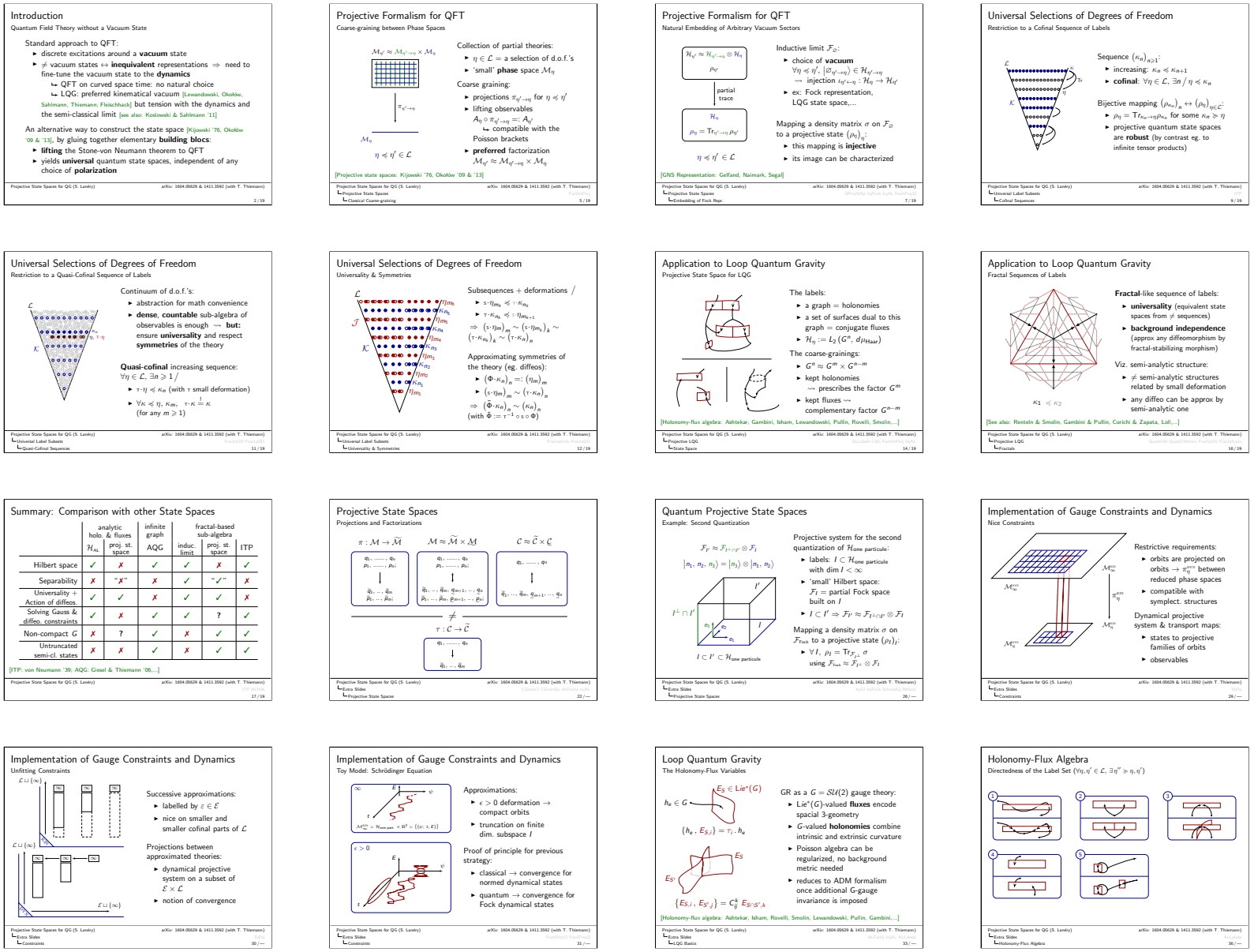
One can construct a quadratic operator Q_N from:

- $\mathrm{h}_{e_1}, \ldots, \mathrm{h}_{e_N}$
- $\mathrm{E}_{S_1}, \ldots, \mathrm{E}_{S_N}$

such that $\langle \mathrm{Q}_N \rangle \leqslant A + B$

But Heisenberg inequalities $\Rightarrow$ **lower** bound on $\langle \mathrm{Q}_N \rangle$ that **diverges** with N!

[See also: Varadarajan; Lewandowski, Okołów, Sahlmann, Thiemann; Fleischhack;...]


Introduction
Quantum Field Theory without a Vacuum State
Standard approach to QFT:
discrete excitations around a vacuum state
≠ vacuum states ↔ inequivalent representations ⇒ need to fine-tune the vacuum state to the dynamics
QFT on curved space time: no natural choice
LQG: preferred kinematical vacuum [Lewandowski, Okołów, Sahlmann, Thiemann; Fleischhack] but tension with the dynamics and the semi-classical limit [see also: Koslowski & Sahlmann '11]
An alternative way to construct the state space [Kijowski '76, Okołów '09 & '13], by gluing together elementary building blocs:
lifting the Stone-von Neumann theorem to QFT
yields universal quantum state spaces, independent of any choice of polarization
Projective Formalism for QFT
Coarse-graining between Phase Spaces
Collection of partial theories:
η ≺ ℒ = a selection of d.o.f.'s
'small' phase space $\mathcal{M}_\eta$
Coarse graining:
projections $\pi_{\eta'\to\eta}$ for $\eta \prec \eta'$
lifting observables
$A_\eta \circ \pi_{\eta'\to\eta} =: A_{\eta'}$
compatible with the Poisson brackets
preferred factorization
$\mathcal{M}_{\eta'} \approx \mathcal{M}_{\eta'\to\eta} \times \mathcal{M}_\eta$
[Projective state spaces: Kijowski '76, Okołów '09 & '13]
Projective Formalism for QFT
Natural Embedding of Arbitrary Vacuum Sectors
Inductive limit $\mathcal{F}_{\oplus}$:
choice of vacuum
$\forall \eta \prec \eta', \left|\Omega_{\eta'\to\eta}\right\rangle \in \mathcal{H}_{\eta'\to\eta}$
⇝ injection $\iota_{\eta'\to\eta} : \mathcal{H}_\eta \to \mathcal{H}_{\eta'}$
ex: Fock representation, LQG state space,...
partial trace
Mapping a density matrix σ on $\mathcal{F}_{\oplus}$ to a projective state $(\rho_\eta)_\eta$:
this mapping is injective
its image can be characterized
[GNS Representation: Gelfand, Naimark, Segal]
Universal Selections of Degrees of Freedom
Restriction to a Cofinal Sequence of Labels
Sequence $(\kappa_n)_{n\geq 1}$:
increasing: $\kappa_n \prec \kappa_{n+1}$
cofinal: $\forall \eta \in \mathcal{L}, \exists n / \eta \prec \kappa_n$
Bijective mapping $(\rho_{\kappa_n})_n \leftrightarrow (\rho_\eta)_{\eta\in\mathcal{L}}$:
$\rho_\eta = \mathrm{Tr}_{\kappa_n\to\eta}\rho_{\kappa_n}$ for some $\kappa_n \succcurlyeq \eta$
projective quantum state spaces are robust (by contrast eg. to infinite tensor products)
Universal Selections of Degrees of Freedom
Restriction to a Quasi-Cofinal Sequence of Labels
Continuum of d.o.f.'s:
abstraction for math convenience
dense, countable sub-algebra of observables is enough ⇝ but: ensure universality and respect symmetries of the theory
Quasi-cofinal increasing sequence:
$\forall \eta \in \mathcal{L}, \exists n \geqslant 1 /$
$\tau\cdot\eta \prec \kappa_n$ (with τ small deformation)
$\forall \kappa \prec \eta, \kappa_m, \tau\cdot\kappa \overset{\perp}{=} \kappa$ (for any $m \geqslant 1$)
Universal Selections of Degrees of Freedom
Universality & Symmetries
Subsequences + deformations /
$s\cdot\eta_{m_k} \prec \tau\cdot\kappa_{n_k}$
$\tau\cdot\kappa_{n_k} \prec s\cdot\eta_{m_{k+1}}$
$\Rightarrow (s\cdot\eta_m)_m \sim (s\cdot\eta_{m_k})_k \sim (\tau\cdot\kappa_{n_k})_k \sim (\tau\cdot\kappa_n)_n$
Approximating symmetries of the theory (eg. diffeos):
$(\Phi\cdot\kappa_n)_n =: (\eta_m)_m$
$(s\cdot\eta_m)_m \sim (\tau\cdot\kappa_n)_n$
$\Rightarrow (\tilde{\Phi}\cdot\kappa_n)_n \sim (\kappa_n)_n$ (with $\tilde{\Phi} := \tau^{-1}\circ s\circ\Phi$)
Application to Loop Quantum Gravity
Projective State Space for LQG
The labels:
a graph = holonomies
a set of surfaces dual to this graph = conjugate fluxes
$\mathcal{H}_\eta := L_2(G^n, d\mu_{\mathrm{Haar}})$
The coarse-grainings:
$G^n \approx G^m \times G^{n-m}$
kept holonomies ⇝ prescribes the factor G^m
kept fluxes ⇝ complementary factor G^{n-m}
[Holonomy-flux algebra: Ashtekar, Gambini, Isham, Lewandowski, Pullin, Rovelli, Smolin,...]
Application to Loop Quantum Gravity
Fractal Sequences of Labels
Fractal-like sequence of labels:
universality (equivalent state spaces from ≠ sequences)
background independence (approx any diffeomorphism by fractal-stabilizing morphism)
Viz. semi-analytic structure:
≠ semi-analytic structures related by small deformation
any diffeo can be approx by semi-analytic one
[See also: Rentelń & Smolin, Gambini & Pullin, Corichi & Zapata, Loll,...]
Summary: Comparison with other State Spaces
analytic holo. & fluxes | infinite graph | fractal-based sub-algebra
$\mathcal{H}_{AL}$ | proj. st. space | AQG | induc. limit | proj. st. space | ITP
Hilbert space ✓ ✗ ✓ ✓ ✗ ✓
Separability ✗ "✗" ✗ ✓ "✓" ✗
University + Action of diffeos. ✓ ✓ ✗ ✓ ✓ ✗
Solving Gauss & diffeo. constraints ✓ ✗ ✓ ✓ ? ✓
Non-compact G ✗ ? ✓ ✗ ✓ ✓
Untruncated semi-cl. states ✗ ✗ ✓ ✗ ✓ ✓
[ITP: von Neumann '39; AQG: Giesel & Thiemann '06,...]
Projective State Spaces
Projections and Factorizations
$\pi : \mathcal{M} \to \widetilde{\mathcal{M}}$
$\mathcal{M} \approx \widetilde{\mathcal{M}} \times \underline{\mathcal{M}}$
$\mathcal{C} \approx \widetilde{\mathcal{C}} \times \underline{\mathcal{C}}$
$\tau : \mathcal{C} \to \widetilde{\mathcal{C}}$
Quantum Projective State Spaces
Example: Second Quantization
$\mathcal{F}_{I'} \approx \mathcal{F}_{I\cap I'} \otimes \mathcal{F}_I$
$|n_1, n_2, n_3\rangle = |n_3\rangle \otimes |n_1, n_2\rangle$
Projective system for the second quantization of $\mathcal{H}_{\text{one particle}}$:
labels: $I \subset \mathcal{H}_{\text{one particle}}$ with dim $I < \infty$
'small' Hilbert space: $\mathcal{F}_I$ = partial Fock space built on I
$I \subset I' \Rightarrow \mathcal{F}_{I'} \approx \mathcal{F}_{I^\perp\cap I'} \otimes \mathcal{F}_I$
Mapping a density matrix σ on $\mathcal{F}_{\mathrm{Fock}}$ to a projective state $(\rho_I)_I$:
$\forall I,\ \rho_I = \mathrm{Tr}_{\mathcal{F}_{I^\perp}}\sigma$ using $\mathcal{F}_{\mathrm{Fock}} \approx \mathcal{F}_{I^\perp} \otimes \mathcal{F}_I$
Implementation of Gauge Constraints and Dynamics
Nice Constraints
Restrictive requirements:
orbits are projected on orbits → π_η^{ACG} between reduced phase spaces
compatible with symplect. structures
Dynamical projective system & transport maps:
states to projective families of orbits
observables
Implementation of Gauge Constraints and Dynamics
Unfitting Constraints
Successive approximations:
labelled by $\varepsilon \in \mathcal{E}$
nice on smaller and smaller cofinal parts of $\mathcal{L}$
Projections between approximated theories:
dynamical projective system on a subset of $\mathcal{E} \times \mathcal{L}$
notion of convergence
Implementation of Gauge Constraints and Dynamics
Toy Model: Schrödinger Equation
Approximations:
$\epsilon > 0$ deformation → compact orbits
truncation on finite dim. subspace I
Proof of principle for previous strategy:
classical → convergence for normed dynamical states
quantum → convergence for Fock dynamical states
Loop Quantum Gravity
The Holonomy-Flux Variables
$h_e \in G$
$E_S \in \mathrm{Lie}^*(G)$
$\{h_e, E_{S,i}\} = \tau_i . h_e$
$\{E_{S,i}, E_{S',j}\} = C_{ij}^k E_{S\cap S',k}$
GR as a $G = SU(2)$ gauge theory:
$\mathrm{Lie}^*(G)$-valued fluxes encode spacial 3-geometry
G-valued holonomies combine intrinsic and extrinsic curvature
Poisson algebra can be regularized, no background metric needed
reduces to ADM formalism once additional G-gauge invariance is imposed
[Holonomy-flux algebra: Ashtekar, Isham, Rovelli, Smolin, Lewandowski, Pullin, Gambini,...]
Holonomy-Flux Algebra
Directedness of the Label Set $(\forall \eta, \eta' \in \mathcal{L}, \exists \eta'' \succcurlyeq \eta, \eta')$