

Electromagnetic fields with vanishing scalar invariants¹

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Contents

- 1 Null electromagnetic fields
- 2 Null gravitational fields
- 3 Main result: VSI p -forms
- 4 Universal solutions

Null electromagnetic fields

Definition [Synge (Silberstein, Bateman, Rainich, Ruse, ...)]

- ① $F_{ab}F^{ab} = 0 \quad (\Leftrightarrow E^2 - B^2 = 0)$
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- “ ϵ -property”
- solutions of Maxwell \Rightarrow also solve Born-Infeld's and any NLE!
[Schrödinger'35,'43]

Null gravitational fields

All Riemann scalar invariants $I(\text{Riem})$ vanish:

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- in HD: **Einstein+type N** \Rightarrow also solve

quadratic (\supset Gauss-Bonnet): $G_{ab} + \Lambda_0 g_{ab} + (\text{Riem})^2 + \nabla^2 \text{Ric} = 0$

Lovelock gravity: $G_{ab} + \Lambda_0 g_{ab} + \sum_{k=2} (\text{Riem})^k = 0$

[Pravdová-Pravda'08, Málek-Pravda'11, Reall-Tanahashi-Way'14]

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these are in fact **VSI** spacetime: any $I(\text{Riem}, \nabla \text{Riem}, \dots) = 0!$

[Pravda-Pravdová-Coley-Milson'02, Coley-Milson-Pravda-Pravdová'04]

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$$\mathcal{L} = \mathcal{L}(F, \nabla F, \dots)$$

old idea [Bopp'40, Podolsky'42]

effective theories motivated by string theory

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We studied this in arbitrary dimension n for a p -form F .

Main result: VSI p -forms

Theorem ([M.O.-Pravda'15])

The following two conditions are equivalent:

- ① *a p -form field \mathbf{F} is **VSI** in a spacetime g_{ab}*
- ②
 - (a) *$\mathbf{F} = \ell \wedge \omega$, with $\ell_a \ell^a = 0 = \ell^{a_1} \omega_{a_1 \dots a_{p-1}}$ (i.e., VSI_0)*
 - (b) *$\mathcal{L}_\ell \mathbf{F} = 0$*
 - (c) *g_{ab} is “degenerate Kundt”.*

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 - (2c) $\Leftrightarrow \ell$ is geodesic, shearfree, twistfree, expansionfree and all $\nabla^k \text{Riem}$ are “multiply aligned” with ℓ
(includes all Einstein-Kundt, Minkowski, (A)dS, ...)
- [Coley-Hervik-Pelavas'09, Coley-Hervik-Papadopoulos-Pelavas'09]

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- proof based on “algebraic VSI theorem” [Hervik'11] and boost-weight classification [Milson-Coley-Pravda-Pravdová'05]

Adapted coordinates ($\ell = \partial_r$, $\ell_a dx^a = du$)

$$ds^2 = 2du[dr + H(u, r, x)du + W_\alpha(u, r, x)dx^\alpha] + g_{\alpha\beta}(u, x)dx^\alpha dx^\beta$$

$$\mathbf{F} = \frac{1}{(p-1)!} f_{\alpha_1 \dots \alpha_{p-1}}(u, x) du \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{p-1}}$$

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- waves in Minkowski, (A)dS, Nariai, gyratons, ...

Examples of VSI tensors:

²[M.O.-Pravda'15]

³[Pravda-Pravdová-Coley-Milson'02, Coley-Milson-Pravda-Pravdová'04,
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Examples of VSI tensors:

	vector ℓ	p -form F^2	Riem ³
VSI ₀	null	N	III (N,O)
VSI ₁	Kundt	N, $\mathcal{L}_\ell F = 0$, Kundt	N, $\kappa = 0$, $\sigma\Psi_4 = \rho\Phi_{22}$
VSI ₂	Kundt, Riem II	N, $\mathcal{L}_\ell F = 0$, Kundt, Riem II	III, Kundt
VSI ₃	degKundt	N, $\mathcal{L}_\ell F = 0$, degKundt	“
\vdots	“	“	“
VSI	“	“	“

²[M.O.-Pravda'15]³[Pravda-Pravdová-Coley-Milson'02, Coley-Milson-Pravda-Pravdová'04, Pelavas-Coley-Milson-Pravda-Pravdová'05]

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Can some VSI F solve any theory $\mathcal{L} = \mathcal{L}(F, \nabla F, \dots)$?

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- Born-Infeld NLE: solved by any null Maxwell field

$$F_{[ab,c]} = 0, \quad \left(\frac{F^{ab} - \mathcal{G}^* F^{ab}}{\sqrt{1 + \mathcal{F} - \mathcal{G}^2}} \right)_{;b} = 0$$

$$(\mathcal{F} \equiv \frac{1}{2} F_{ab} F^{ab}, \mathcal{G} \equiv \frac{1}{4} F_{ab} {}^* F^{ab}) \text{ [Schrödinger'35,'43]}$$

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- arbitrary $\mathcal{L} = \mathcal{L}(F, \nabla F, \dots)$: any VSI F in type III Kundt
- coupling to gravity also possible: further restrictions
cf. also [Güven’87, Horowitz-Steif’90, Coley’02]

NLE in [Kichenassamy’59, Kremer-Kichenassamy’60, Peres’60]

An Einstein-Maxwell universal example ($n = 4$, $p = 2$):

$$ds^2 = 2du \left[dr + \frac{1}{2} (xr - xe^x - 2\kappa_0 e^x c^2(u)) du \right] + e^x (dx^2 + e^{2u} dy^2)$$

$$\mathbf{F} = e^{x/2} c(u) du \wedge \left(-\cos \frac{ye^u}{2} dx + e^u \sin \frac{ye^u}{2} dy \right)$$

- $\ell = \partial_r$ is Kundt and *recurrent* ($\ell_{a;b} = f \ell_a \ell_b$)
- \mathbf{F} is VSI
- Petrov type III
- obtained by “charging” a vacuum metric of Petrov [Petrov'62]