Uniqueness of the Fock quantization of Dirac fields with unitary dynamics

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Motivation

- Dirac (fermion) fields describe realistic matter contents in Physics: cosmology, condensed matter (e.g. graphene)...

- Infinite ambiguity in their quantum description:
  - Choice of (Fock) representation of the CAR's.

- In non-stationary geometries, it is reasonable to require the dynamics to be unitary (quantum coherence).

- Additional criterion: invariant vacuum under the spatial isometries.

- **Result:** unique Fock representation of the CAR's.

Closed FLRW cosmology:
Invariant vacua

Cosmological model

- FLRW cosmology, scale factor $\exp[\alpha(\eta)]$, spatial sections $\approx S^3$.

- Minimally coupled massive Dirac field, described by

  $\phi_A, \bar{\chi}_{A'} \qquad A=1,2;\ A'=1',2', \qquad$ <u>Grassmann</u> variables.

- Mode decomposition in terms of Dirac operator eigenspinors on S^3:

$$\left.\begin{array}{l} \phi_A \longrightarrow m_{np}(\eta), \bar{r}_{np}(\eta), \text{ eigenvalues } \pm\omega_n \\ \bar{\chi}_{A'} \longrightarrow t_{np}(\eta), \bar{s}_{np}(\eta), \text{ eigenvalues } \pm\omega_n \end{array}\right\} \begin{array}{l} \omega_n \sim n \in \mathbb{N} \\ p=1,\ldots,g_n \sim n^2 \end{array}$$

- Fock representation $\longrightarrow$ complex structure on space of initial data.

Isometry invariant vacua

- Isometry group $SO(4)$ $\Longrightarrow$ $\mathrm{Spin}(4)$ on spinors of a chirality.

- Known relation between the Dirac operator eigenspaces on S^3 and the (unitary) irreducible representations of $\mathrm{Spin}(4)$.

- **Result:** any $\mathrm{Spin}(4)$-invariant Fock vacuum is characterized by particle annihilation and antiparticle creation variables:

$$a_{np}(\eta)=f_1^n(\eta)\,x_{np}(\eta)+f_2^n(\eta)\,\bar{y}_{np}(\eta),\qquad (x_{np},y_{np})=\begin{cases}(m_{np},s_{np})\\ \\ (t_{np},r_{np})\end{cases}$$

$$b^{\dagger}_{np}(\eta)=g_1^n(\eta)\,x_{np}(\eta)+g_2^n(\eta)\,\bar{y}_{np}(\eta),$$

$$|f_1^n|^2+|f_2^n|^2=e^{3\alpha}=|g_1^n|^2+|g_2^n|^2,\qquad f_1^n\bar{g}_1^n+f_2^n\bar{g}_2^n=0.$$

Unitary dynamics

Fermion dynamics

- First order Dirac equations, with $' := d/d\eta$

$$x_{np}' = \left(i\omega_n - \frac{3\alpha'}{2} \right) x_{np} - ime^{\alpha}\bar{y}_{np}, \qquad \bar{y}_{np}' = -\left(i\omega_n + \frac{3\alpha'}{2} \right) \bar{y}_{np} - ime^{\alpha} x_{np}.$$

- Same second order equation for all the modes.
  - Known asymptotic behavior of its two independent solutions.

$$\Downarrow$$

- The (relevant) asymptotics of the evolution is known

$$x_{np}(\eta) = \boldsymbol{A}_n(\eta, \eta_0)\, x_{np}(\eta_0) + \boldsymbol{B}_n(\eta, \eta_0)\, \bar{y}_{np}(\eta_0),$$

$$\bar{y}_{np}(\eta) = \bar{\boldsymbol{A}}_n(\eta, \eta_0)\, \bar{y}_{np}(\eta_0) - \bar{\boldsymbol{B}}_n(\eta, \eta_0)\, x_{np}(\eta_0).$$

Unitary dynamics

- Fermion dynamics $\longrightarrow$ time-dependent Bogoliubov transformation:

$$a_{np}(\eta)=\alpha_n^f(\eta,\eta_0)a_{np}(\eta_0)+\beta_n^f(\eta,\eta_0)b_{np}^\dagger(\eta_0)$$

$$b_{np}^\dagger(\eta)=\alpha_n^g(\eta,\eta_0)b_{np}^\dagger(\eta_0)+\beta_n^g(\eta,\eta_0)a_{np}(\eta_0)$$

- The transformation is implementable as a unitary operator in the Fock space defined by $\{a_{np}(\eta_0), b_{np}^\dagger(\eta_0)\}$ if and only if

$$\sum_n g_n|\beta_n^f(\eta,\eta_0)|^2<\infty, \quad \sum_n g_n|\beta_n^g(\eta,\eta_0)|^2<\infty, \qquad \forall\,\eta.$$

- We know that $\omega_n \sim n \in \mathbb{N}$ and $g_n \sim n^2$.

Unitary dynamics: conditions

$$a_{np}(\eta)=f_1^n(\eta)\,x_{np}(\eta)+f_2^n(\eta)\,\bar{y}_{np}(\eta),$$

$$b^{\dagger}_{np}(\eta)=g_1^n(\eta)\,x_{np}(\eta)+g_2^n(\eta)\,\bar{y}_{np}(\eta).$$

- **Result:** Unitarily implementable dynamics if and only if:

$$\left(\begin{array}{ll} f_1^n=\dfrac{me^{5\alpha/2}}{2\omega_n}e^{iF^n}+\vartheta^n_{f,1}, & g_1^n=\bar{f}_2^n e^{iG^n}, \\ f_2^n=e^{iF^n}\sqrt{e^{3\alpha}-|f_1^n|^2}, & g_2^n=-\bar{f}_1^n e^{iG^n}, \end{array}\right) \qquad \forall\,\eta,$$

where $\vartheta^n_{f,1}$ is square summable, for an infinite subset of $\mathbb{N}$, whereas $f_1^n\leftrightarrow g_1^n$, $f_2^n\leftrightarrow g_2^n$, for the complementary subset.

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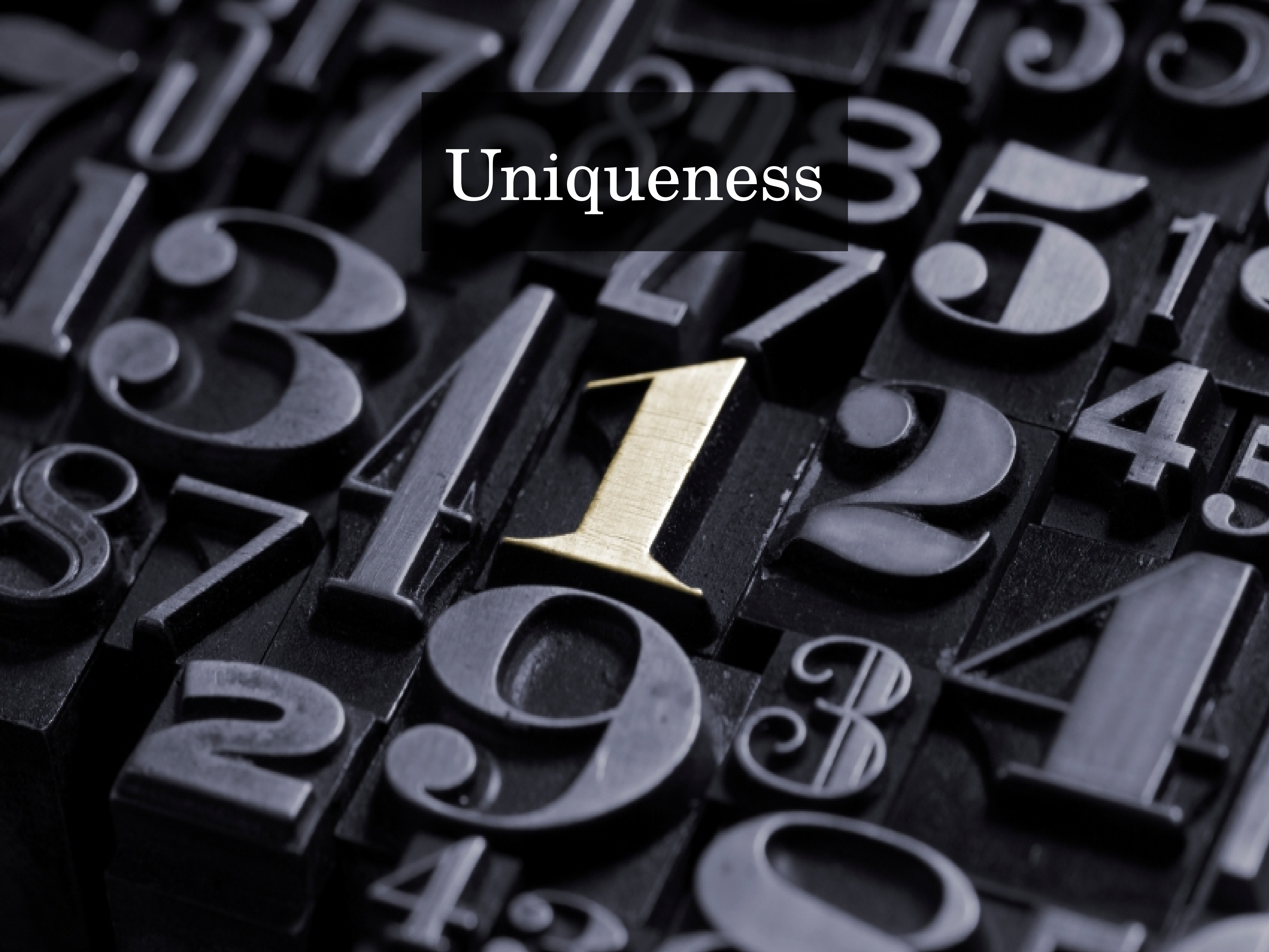
$\Downarrow$

With respect to the first subset, particles $\leftrightarrow$ antiparticles.


Uniqueness

Uniqueness

- Reference representation that admits unitary dynamics:

$$f_1^n = \frac{m e^{5\alpha/2}}{2\omega_n}, \quad f_2^n = \sqrt{e^{3\alpha} - (f_1^n)^2}, \quad g_1^n = f_2^n, \quad g_2^n = -f_1^n, \qquad \forall\, n \in \mathbb{N}.$$

- Any other with the same convention of particles and antiparticles:

$$\tilde{f}_1^n = \frac{m e^{5\alpha/2}}{2\omega_n} e^{i\tilde{F}^n} + \vartheta^n_{\tilde{f},1}, \qquad \sum_{n\in\mathbb{N}} g_n \left|\vartheta^n_{\tilde{f},1}\right|^2 < \infty, \qquad \forall\, n \in \mathbb{N}, \qquad \text{etc.}$$

- Bogoliubov transformation between them:

$$\tilde{a}_{np} = \kappa_n^f a_{np} + \lambda_n^f b_{np}^\dagger, \qquad \lambda_n^f = \vartheta^n_{\tilde{f},1} + O\left(\omega_n^{-2}\right),$$

$$\tilde{b}_{np}^\dagger = \kappa_n^g b_{np}^\dagger + \lambda_n^g a_{np}, \qquad \lambda_n^g = -\bar{\vartheta}^n_{\tilde{f},1} e^{i\tilde{G}_n} + O\left(\omega_n^{-2}\right)$$

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$$\omega_n \sim n \in \mathbb{N}, \quad g_n \sim n^2$$

Conclusions

- Combined criteria of invariance of the vacuum under the isometry group of the closed FLRW cosmology + unitary implementation of the dynamics $\longrightarrow$ unique Fock quantization of the Dirac field.

- Uniqueness attained given a convention of particles and antiparticles.

- The part of the dynamics that can be unitarily implementable is uniquely determined $\longrightarrow$ extraction of explicitly time-dependent functions from the dominant parts of the field.

- Generalization to flat spatial sections, and $2+1$ scenarios.