The conformally invariant wave equation near the cylinder at spacelike infinity

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joint work with

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Introduction

- Conformal compactification of asymptotically simple spacetimes: Mathematical difficulties can arise near spacelike infinity i^0.
- Helmut Friedrich's generalised conformal field equations are adapted to an appropriate treatment of i^0, which was thought of before as a point, but is now 'blown up' to a cylinder $S^2 \times [-1, 1]$ that connects past and future null infinity $\mathscr{I}^{\pm}$.
- To what extend can fields near i^0 be obtained numerically?
- *Toy model: Conformally invariant wave equation* (in 4 dimensions) on Minkowski or Schwarzschild background,

$$g^{ab}\nabla_a\nabla_b f - \frac{R}{6} f = 0.$$

  Conformal weight is -1, i.e. if $\tilde{f}$ solves the equation for a metric $\tilde{g}_{ab}$, then $f = \Theta^{-1}\tilde{f}$ solves the equation for $g_{ab} = \Theta^2\tilde{g}_{ab}$.
- Typical problem in some numerical methods: Time step restricted by CFL condition. $\rightarrow \mathscr{I}^+$ cannot be reached with a finite number of steps. Here: fully pseudospectral time evolution

The fully pseudospectral scheme

- *Basic idea for solving* $1+1$ *dimensional problems:* [JH and Ansorg 2009]

1. Map the physical domain to one (or several) unit square(s) by introducing spectral coordinates $(\sigma,\ \tau) \in [0, 1] \times [0, 1]$ such that the surface on which initial data are given corresponds to $\tau = 0$.
2. Enforce the initial conditions by expressing the unknown function $f(\sigma, \tau)$ in terms of another unknown $f_2(\sigma, \tau)$ via

$$f(\sigma,\tau) = \begin{cases} f_0(\sigma) + \tau f_1(\sigma) + \tau^2 f_2(\sigma,\tau), & \text{if function values and first derivatives are given} \\ f_0(\sigma) + \tau f_2(\sigma,\tau), & \text{if only function values are given} \end{cases}$$

depending on the type of problem (first/second order equations, characteristic/Cauchy initial value problem).

The fully pseudospectral scheme

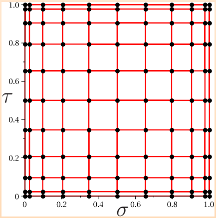
3. Approximate the new unknown f_2 via

$$f_2(\sigma,\tau) \approx \sum_{j=0}^{N_\sigma}\sum_{k=0}^{N_\tau} c_{jk} T_j(2\sigma-1) T_k(2\tau-1)$$

for given spectral resolution (number of polynomials) $n_\sigma \equiv N_\sigma + 1$ and $n_\tau \equiv N_\tau + 1$.

4. Obtain an algebraic system of equations by requiring that the equation(s), and suitable boundary or regularity conditions, are satisfied at a set of collocation points. We choose Gauss-Lobatto nodes $(\sigma_j, \tau_k), j = 0, \ldots, N_\sigma, k = 0, \ldots, N_\tau$:

$$\sigma_j = \sin^2\left(\frac{\pi j}{2N_\sigma}\right), \quad \tau_k = \sin^2\left(\frac{\pi k}{2N_\tau}\right)$$



5. Starting from some initial guess, solve this system iteratively with the Newton-Raphson method.

Minkowski background

- Minkowski metric $\tilde{g}$ in spherical coordinates

$$\tilde{g} = \mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{r}^2 - \tilde{r}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

- Compactification:

$$\tilde{u} = \tilde{t}-\tilde{r},\ \tilde{v} = \tilde{t}+\tilde{r};\quad \tilde{u} = \tan u,\ \tilde{v} = \tan v,\quad T = v+u,\ R = v-u$$

$$T = 2t\cos(r),\quad R = 2r$$

$$g = \mathrm{d}t^2 - 2t\tan r\,\mathrm{d}t\mathrm{d}r - [1+(1-t^2)\tan^2 r]\mathrm{d}r^2 - \sin^2 r(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$$

$$\Theta = \frac{\cos(t\cos r - r)\cos(t\cos r + r)}{\cos r},\quad \tilde{g} = \Theta^{-2} g$$

- *Conformally invariant wave equation:*

$$(1 - t^2\sin^2 r)f_{,tt} - 2t\sin r\cos r\, f_{,tr} - \cos^2 r\, f_{,rr}$$
$$-t(2+\cos^2 r)f_{,t} - 2\frac{\cos^3 r}{\sin r}f_{,r} + \cos^2 r\, f = 0$$

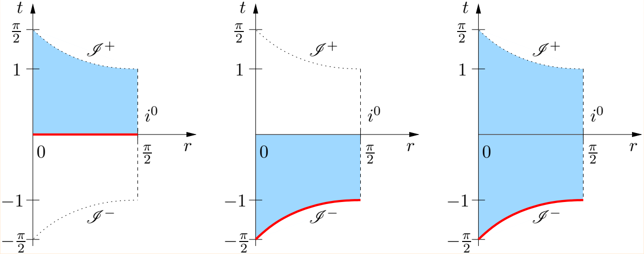
- *Regularity conditions:*

  at $r = 0$: $f_{,r} = 0$, at $r = \frac{\pi}{2}$ (i^0): $f_{,t} = 0$

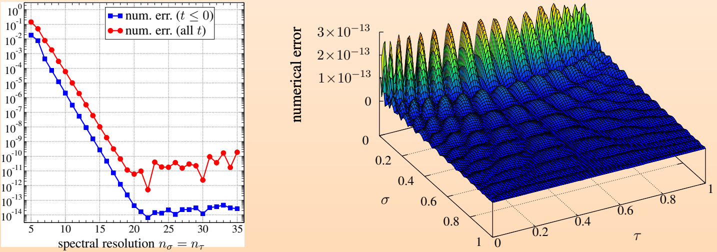
- *General (regular) solution:* $f(t,r) = \dfrac{F(t\cos r - r) - F(t\cos r + r)}{\sin(r)}$

Minkowski background

- *Cauchy or characteristic initial value problems:*



- *Example:* characteristic IVP, $F = \frac{1}{10}x^3$ [Frauendiener and JH 2014]

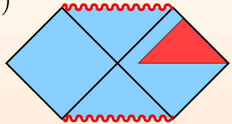


Schwarzschild background

Solution in a neighbourhood of the cylinder at i^0

- Schwarzschild metric $\tilde{g}$ in isotropic coordinates

$$\tilde{g} = -\left(\frac{1-\frac{2m}{\tilde{r}}}{1+\frac{2m}{\tilde{r}}}\right)\mathrm{d}t^2 - \left(1+\frac{m}{2\tilde{r}}\right)^4(\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\sigma^2)$$



- New coordinates $0 < \rho < \rho_{\max} < 1$, $0 \le \tau < 1$ [Friedrich, 2004]:

$$r = \frac{m}{2\tilde{r}},\ t = \frac{2\tilde{t}}{m};\quad t = \int_r^\rho \frac{\mathrm{d}s}{F(s)},\ r = \rho(1-\tau),\ F(r) = \frac{r^2(1-r)}{(1+r)^3}$$

$$g = \frac{2\rho A}{r^2}\mathrm{d}\rho\mathrm{d}\tau - \frac{A}{r^2}[2(1-\tau)-A]\mathrm{d}\rho^2 - \mathrm{d}\sigma^2,\quad A := \frac{F(r)}{F(\rho)},\quad \Theta = \frac{m(1+r)^2}{2r}$$

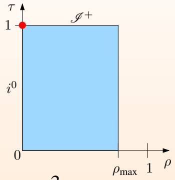
$\tau = 1$: $\mathscr{I}^+$, $\rho = 0$: i^0, $\rho = 1$: coordinate singularity (A diverges)

- *Conformally invariant wave equation:*

$$[2(1-\tau)-A]f_{,\tau\tau} + 2\rho f_{,\tau\rho} - 2\left[1 - \frac{\rho(1-2r)}{r(1-r^2)}A\right]f_{,\tau} + 4\frac{\rho^2 A}{r(1+r)^2}f = 0$$

Schwarzschild background

Solution in a neighbourhood of the cylinder at i^0



- *General behaviour of f near i^0:*

  Expansion:

$$f(\tau,\rho) = \phi_0(\tau) + \rho\phi_1(\tau) + \rho^2\phi_2(\tau) + \rho^3\phi_3(\tau) + \dots$$

  Conformal wave eq. leads to ODEs for ϕ_0, ϕ_1, ...

  1. $\phi_0(\tau) = c_0 + c_1 \ln\frac{1-\tau}{1+\tau}$ $\rightarrow$ choose data with $f_{,\tau}(0,0) = -2c_1 = 0$
  2. $\phi_1(\tau) = -2c_0\tau - \frac{2c_0+c_2}{1+\tau} + c_3$ $\rightarrow$ regular
  3. $\phi_2(\tau) = \frac{4(2c_0-c_3)}{1+\tau)^2}(1-\tau)^2\ln(1-\tau)$ + regular terms
     $\rightarrow f_{,\rho\rho\tau\tau}$ singular, unless $(f_{,\rho} + f_{,\tau\rho})(0,0) = c_3 - 2c_0 = 0$
  4. $\phi_3(\tau) = \frac{8(6c_0+5c_2+3c_4)}{3(1+\tau)^2}(1-\tau)^3\ln(1-\tau)$ + regular terms
     $\rightarrow f_{,\rho\rho\rho\tau\tau\tau}$ singular, unless $(3f_{,\rho\rho} + 14f_{,\rho} - 36f)(0,0) = 0$

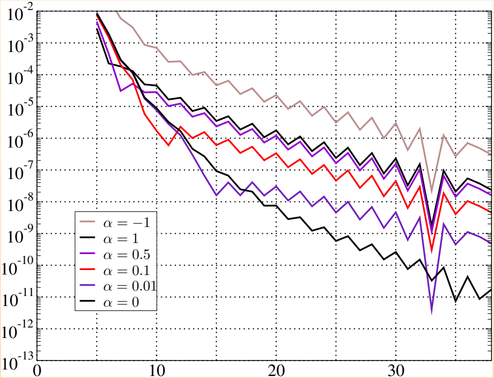
  $\vdots$

Schwarzschild background

Solution in a neighbourhood of the cylinder at i^0

Family of initial data at $\tau = 0$: $f = 2 + \sin(5\rho)$, $f_{,\tau} = (\alpha - 5)\rho$

- Choose $\rho_{\max} = 0.5$
- For all α: ϕ_0 regular
- For $\alpha = 0$: ϕ_2 regular



$\rightarrow$ algebraic convergence

Schwarzschild background

Including the horizon

- Again starting from isotropic coordinates, perform the following, modified coordinate transformation (with parameter $d > \frac{1}{2}$):

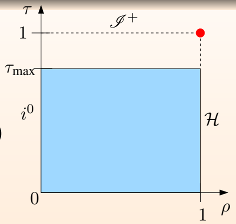
$$t = \int_{r}^{\rho(2-\rho)} \frac{\mathrm{d}s}{F(s)}, \quad r = \frac{\rho(1-\tau)}{w+\rho(1-\tau)}; \quad w = \frac{1-\rho}{(2-\rho)(1+d\rho)}, \quad F(r) = \frac{r^2(1-r)}{(1+r)^3}$$

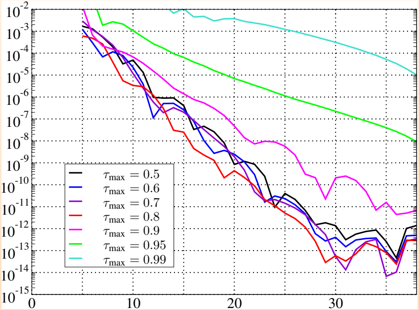
- *Boundaries:* $\mathscr{I}^+: \ \tau = 1, \quad i^0: \ \rho = 0, \quad$ horizon $\mathcal{H}: \ \rho = 1$
- Similar behaviour at the cylinder (conditions on initial data can remove the leading-order logarithmic singularities at $\rho = 0$, $\tau = 1$)

Schwarzschild background

Including the horizon

- *Numerical example:*
  Domain: $0 \leq \rho \leq 1$, $0 \leq \tau_{\text{max}} < 1$
  Parameter: $d = 2$
  Initial data: $f = 2 + \sin(5\rho), f_{,\tau} = -5\rho$ at $\tau = 0$





→ spectral convergence and highly accurate solutions, if the numerical domain is not too close to the singularity at $\rho = 1$, $\tau = 1$

[Frauendiener and JH, in preparation]

Summary

- *Minkowski background:*
  - For initial data subject to regularity conditions (at the intersection of the initial surface with i^0 and at the origin) the conformally invariant wave equation has solutions that are regular up to $\mathscr{I}^+$.
  - We can solve Cauchy problems in a neighbourhood of the cylinder or globally, and characteristic initial value problems with data at $\mathscr{I}^-$.

    $\rightarrow$ spectral convergence and highly accurate solutions
- *Schwarzschild background:*
  - Solutions generally have logarithmic singularities at the intersection of $\mathscr{I}^+$ and i^0. The leading singularities can be removed with appropriate restrictions for the initial data.

    $\rightarrow$ algebraic convergence, still very accurate solutions
  - We can use coordinates that include the event horizon $\mathcal{H}$ and solve the wave equation in a domain with $0 \le \tau \le \tau_{\max} < 1$ avoiding $\mathscr{I}^+$.

    $\rightarrow$ spectral convergence, if $\tau_{\max}$ is not too close to 1